ESTIMATION OF SCALE PARAMETER OF EXPONENTIAL AND TRUNCATED EXPONENTIAL DISTRIBUTIONS IN EXCHANGEABLE OUTLIER MODELS

by

SHUBHA RANI

TH
MATH/1989/D
Sh19e

ATH
389
D
RAN
EST

DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR

JUNE 1989

ESTIMATION OF SCALE PARAMETER OF EXPONENTIAL AND TRUNCATED EXPONENTIAL DISTRIBUTIONS IN EXCHANGEABLE OUTLIER MODELS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

by

SHUBHA RANI

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

JUNE 1989

To

my father

and

the memory of my mother

MATH-1989-D-RANI-EST

13 JUL 1990

CENTRAL LIBRARY
I. I. T., KANPUR

Acc. No. A.108466

CERTIFICATE

This is to certify that the matter in the thesis entitled "Estimation of Scale Parameter of Exponential and Truncated Exponential Distributions in Exchangeable Outlier Models" by Shubha Rani for the award of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by her under my supervision and guidance. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

P.C. Joshi 9-6-1989

P. C. Joshi
Professor
Department of Mathematics
Indian Institute of Technolo
Kanpur.

June, 1989

ACKNOWLEDGEMENTS

I would like to express my deep sense of gratitude to my thesis supervisor Professor P.C. Joshi for his most valuable suggestions and excellent guidance. He has always been a source of inspiration and the rich experience I gained during this work will be a pleasant memory for ever.

I extend my thanks to all my friends for their constant support and in particular to Vellaisamy, Neeraj, Manjul Somesh, Tanuja, Rajiv and Renu.

Finally, Mr. R.N. Srivastava deserves an appreciation for his great care and patience in typing the thesis.

- Shubha Rani

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION AND SUMMARY

1.1 Scope

This study is mainly concerned with the estimation of scale parameter of exponential distributions and truncated exponential distributions. It is well known that the exponential distribution occurs quite often in life testing situations, for example, see Epstein and Sobel (1953, 1954), Proschan (1963), Bain (1978) etc. A random variable X is said to follow an exponential distribution with scale parameter σ if its probability density function (pdf) $f(x;\sigma)$ is given by

$$f(x;\sigma) = \frac{1}{\sigma} \exp(-x/\sigma) , \qquad x > 0, \ \sigma > 0. \qquad (1.1.1)$$

Suppose we have a sample of n observations such that (n-1) of them have an exponential distribution with pdf $f(x;\sigma)$ and the remaining one comes from another exponential distribution with pdf $f(x; \sigma/\alpha)$, where $\alpha > 0$. No prior information about the outlying observation is available and the probability that any X_i comes from $f(x; \sigma/\alpha)$ is $\frac{1}{n}$. Further, given the index i of the outlying variable, the random variables $X_1,\ldots,X_n$ are independent. Under this semi-Bayesian approach of Kale and Sinha (1971), the joint pdf of $X_1,\ldots,X_n$ is given by

$$f(x_1,\ldots,x_n) = \frac{1}{n} \sum_{i=1}^{n} f(x_i, \ \sigma/\alpha) \prod_{\substack{j=1 \\ j\neq i}}^{n} f(x_j; \sigma)$$

$$= \frac{\alpha}{n\sigma^n} \sum_{i=1}^{n} e^{-\sum_{j=1}^{n} x_j/\sigma} e^{(1-\alpha)x_i/\sigma} . \qquad (1.1.2)$$

It is easy to show that under this model, $X_1,\ldots,X_n$ are exchangeable random variables and consequently this model is called the exchangeable model for a single outlier (Barnett and Lewis, 1984, p. 42). An alternative interpretation for such a model is provided by David and Shu (1978).

An exponential random variable truncated at x_0 has the pdf

$$f(x;\sigma) = \frac{1}{(1-e^{-x_0/\sigma})} e^{-x/\sigma}, \quad 0 < x \leq x_0, \; \sigma > 0. \qquad (1.1.3)$$

In many experiments, it may happen that we cannot observe all possible values of a random variable and so we have samples from a truncated distribution. Cohen (1955), Deemer and Votaw (1955) and Bain and Weeks (1964), among others, have studied the truncated exponential distribution, and have also given some applications of this distribution. Some examples of truncated normal distribution are given in Schneider (1986) where another type of truncated distribution with proportion of truncation known is also discussed. Similar cases for truncated exponential distributions are studied by Saleh et al. (1975), Joshi (1978), Balakrishnan and Joshi (1984) etc.

Some estimators of σ are proposed and the robustness of these estimators when the sample contains outliers under exchangeable model is studied. Most of these estimators are linear combinations of order statistics. Here some recurrence relations for the moments of order statistics are derived for

calculating the biases and mean square errors of estimators. While doing this study, we have also obtained some identities among the moments of order statistics. These may be applied for checking the calculations of moments of order statistics from exponential and truncated exponential distributions, which are evaluated in this study.

The main topics which are included in the thesis are as follows:

1. Recurrence relations among the moments of order statistics from exponential and truncated exponential distributions.
2. Some identities for the moments of order statistics in the general case.
3. Estimation of scale parameter of an exponential distribution in a single outlier exchangeable model.
4. Estimation of scale parameter of an exponential distribution for two outlier exchangeable model.
5. Estimation problems for the truncated exponential distribution.

Some of the existing results are generalized and some new results are obtained. Suitable tables are provided to support the theory.

1.2 Notations

Let X be a random variable having a continuous cumulative distribution function F(x), and probability density function f(x). Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from this distribution, and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be the

corresponding order statistics obtained by rearranging $X_1, \ldots, X_n$ in an increasing order of magnitude. We shall use the following notations and abbreviations wherever possible.

E(X) = mean of X

V(X) = variance of X

mse(T) = mean square error of an estimator T

i.i.d. = independently and identically distributed

cdf = cumulative distribution function

pdf = probability density function

mme = method of moment estimator

ml = maximum likelihood

mle = maximum likelihood estimator

mmle = modified maximum likelihood estimator

$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt, \quad a > 0,\ b > 0$

$I_p(a,b)$ = incomplete beta function

$$= \{B(a,b)\}^{-1} \int_0^p u^{a-1}(1-u)^{b-1}du, \quad a > 0,\ b > 0,\ 0 < p < 1$$

$$= \sum_{i=a}^{a+b-1} \binom{a+b-1}{i} p^i (1-p)^{a+b-1-i}, \tag{1.2.1}$$

when a, b are positive integers

$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt, \quad a > 0$

$E_q(a)$ = incomplete gamma function

$$= \frac{1}{\Gamma(a)} \int_0^q t^{a-1} e^{-t} dt$$

$$= 1 - \sum_{i=0}^{a-1} \frac{e^{-q} q^i}{(i)!}, \quad \text{when a is a positive integer} \tag{1.2.2}$$

$$F_{r:n}(x) = \text{cdf of } X_{(r)}$$
$$= I_{F(x)}(r, n-r+1)$$
$$\nu^{(k)}_{r:n} = E(X^k_{(r)})$$

A sample is said to be homogeneous if all the observations in the sample are i.i.d.

Est. = estimator

wlog = without loss of generality.

1.3 Recurrence relations among moments of order statistics from exponential and truncated exponential distributions

In Section 1.1, we have stated the exponential and truncated exponential distributions. Here first we give some additional examples where truncated distributions occur.

(i). Biological systems may often be presumed to include threshold values beyond which malfunction and death occurs, with the result that such extreme values are never observed in a functional system.

(ii). Feedback systems, also, may generate bounded variables when extreme values provide the signal triggering the feedback mechanism.

(iii). Suppose there is a measuring device which is unable to record the values greater than some point x_o, but the experiment is continued until a fixed number of measurements n are recorded and the number of unrecorded observations is unknown.

These examples give an idea that there are many situations where we have samples from a truncated distribution.

For homogeneous samples from arbitrary distribution, several recurrence relations among the moments of order statistics are given in literature, for example, see Govindarajulu (1963), Arnold (1977), David (1981), etc., for some such relations.

David and Shu (1978) have derived the recurrence relations among the moments of order statistics from an arbitrary distribution when sample contains a single outlier. They have also applied their results to normal distribution. Recently, Balakrishnan and Ambagaspitiya (1988), and Bapat and Beg (1989), have obtained similar results for double exponential and exponential distributions respectively.

In Chapter 2, we derive some recurrence relations among the single and product moments of order statistics from the right truncated exponential distribution in one outlier exchangeable model, i.e., in a sample of size n, (n-1) observations have a truncated exponential distribution with pdf given at equation (1.1.3) and one observation comes from another truncated exponential distribution with pdf $f(x;\sigma/\alpha)$, $\alpha > 0$. In particular, single moments are expressed in terms of lower order moments of order statistics in homogeneous and a single outlier cases. It is shown that one can evaluate all these moments in a systematic manner. Some such relations for the product moments of order statistics are also obtained. Results for the exponential distribution are obtained as a special case. Tables for the means, variances and covariances of order statistics from truncated exponential distribution for $n = 5$ and various α values, are given.

1.4 Some identities for the moments of order statistics in the general case

Moment identities are useful for checking the calculations of moments of order statistics. These are also useful in establishing some combinatorial identities. Several such identities are available in literature, for example, see Govindarajulu (1963), Joshi (1971, 1973), Arnold (1977), David (1981), Joshi and Balakrishnan (1981, 1982), Balakrishnan and Malik (1985), and Balakrishnan (1986) for some such relations. For an up-to-date summary, see, Balakrishnan et al. (1988) and Malik et al. (1988).

In Chapter 3, we consider a homogeneous sample from an arbitrary continuous distribution and order statistics are obtained from this sample. Some new identities among the moments of order statistics are derived. These are more general in nature, and are also applicable when extreme order statistics do not have finite moments. One such identity is

$$\sum_{j=1}^{n} \frac{(n-j+1)}{(n+1)(n+2)} \nu_{j+1:n+2} = \sum_{j=1}^{n} \frac{1}{(j+1)(j+2)} \nu_{2:j+2} ,$$

which, among others, is useful for distributions for which $\nu_{1:n+2}$ and $\nu_{n+2:n+2}$ do not exist.

These identities may be used for checking the computations of moments of order statistics from truncated exponential distribution for $\alpha = 1$. We have also applied these results to some specific distributions to establish some combinatorial identities.

1.5 Estimation of scale parameter of an exponential distribution in a single outlier exchangeable model

There are many situations in which the assumption of i.i.d. random variables is unrealistic. This may happen due to a variety of reasons and in such cases, some observations in the sample are either too large or too small (Barnett and Lewis, 1984). These observations are suspected to be outliers and this situation is labelled as a outlier situation.

In Chapter 4, we consider the estimation of scale parameter of an exponential distribution with pdf $f(x;\sigma)$ given at equation (1.1.1) under the exchangeable model for a single outlier.

Estimation of parameter σ for this model has been considered by many authors. For $0 < \alpha < 1$, Kale and Sinha (1971) considered a class of estimators U_7 which is the linear combination of m smallest order statistics, and is given by

$$U_7 = \frac{1}{(m+1)} \{ \sum_{i=1}^{m-1} X_{(i)} + (n-m+1) X_{(m)} \}, \quad m = 1,\ldots,n.$$

Joshi (1972) gave the distribution theory of order statistics for this case, and has also tabulated the optimum value of m for n = 2(1)10(5)20(10)50 and α = .05(.05)1.00.

Chikkagoudar and Kunchur (1980) proposed another estimator U_3 which is given by

$$U_3 = \sum_{i=1}^{n} [1 - \frac{2i}{n(n+1)}] \frac{X_{(i)}}{n} .$$

They have shown that this estimator is more efficient than U_7 in some cases.

Kimber (1983) has also considered the estimation of scale parameter of exponential distribution as a special case of gamma distribution. Without claiming any estimator to be the best, he suggests to use an estimator based on first n-m order statistics which is of the form

$$U_6 = C \sum_{i=1}^{n-m} X_{(i)} ,$$

where C is the unbiasing factor. In this thesis, we have also included similar estimators like $\sum_{i=1}^{n-1} X_{(i)}/(n-1)$ and $\sum_{i=1}^{n-2} X_{(i)}/(n-2)$. Joshi (1988) reviewed the estimation of σ for $0 < \alpha < 1$ briefly, and discussed moment estimators, modified moment estimators, maximum likelihood estimators and modified maximum likelihood estimators.

Kale (1975) and Gather and Kale (1988), have obtained the maximum likelihood estimators of parameters for labelled model, when the sample may have $k < n$ spurious observations from another distribution. In Section 4.2, the solution of maximum likelihood equations is obtained for the model given at equation (1.1.2).

In Section 4.3, exact expressions of biases and mean square errors are evaluated for various estimators which are useful for estimating σ in this situation. In Section 4.4, the limiting values of biases and mean square errors as $\alpha \to 0$, $\alpha = 1$ and $\alpha \to \infty$ are calculated. Saleh (1966) has considered the use of optimum order statistics for estimation of σ. In Section 4.5, a similar table is given for two and three optimum order statistics, which is useful for the one outlier case.

David and Shu (1978) have studied the robustness of estimators for mean of normal distribution by comparing various estimators. Balakrishnan and Ambagaspitiya (1988) have considered the case of double exponential distribution. Gather (1986) has also studied some such estimators for the exponential distribution by premium-protection method. In Sections 4.6 and 4.7, the robustness of various estimators discussed in Section 4.3 is studied.

1.6 Estimation of scale parameter of an exponential distribution for two outlier exchangeable model

When random variables are independently, but not identically distributed, Vaughan and Venables (1972) have given the density function of order statistics in terms of permanents. Bapat and Beg (1989) have obtained the distribution function as a function of permanents. They have also obtained the moment generating function and moments of order statistics corresponding to independent exponential random variables.

In Chapter 5, the estimation of scale parameter is considered under the exchangeable model when the sample contains two outliers, i.e., if there is a sample of n observations, then (n-2) of these observations are exponentially distributed with pdf $f(x;\sigma)$, given in equation (1.1.1) and the remaining two come from another exponential distribution with pdf $f(x;\sigma/\alpha)$, $\alpha > 0$. This generalization of one outlier case to two outlier case has been also studied by Ranganathan (1981).

In Sections 5.2 and 5.3, the distribution theory of order statistics is discussed. The correlation coefficient bet een the smallest and the largest order statistics is evaluated in Section 5.4. Similar expression for one outlier case is derived by Gross et al. (1986). In Section 5.5, the maximum likelihood estimator of σ is obtained. In the last section, robustness properties of these estimators are investigated by evaluating exact mse and performing some simulation.

1.7 Estimation problems for the truncated exponential distribution

In Chapter 6, we consider the sample from truncated exponential distribution for two cases, viz.,(i) when proportion of truncation on right 1-P is known,and (ii) when truncation point x_o is known with pdf given at equation (1.1.3). In Section 6.2, the joint distribution of $S = \Sigma X_{(i)}$ and $Y = \frac{X_{(n)}}{-\log(1-P)}$ and some related distributions are obtained for case (i). In Section 6.3, mle of σ is derived for case (i) and the exact distribution of mle is also obtained for small sample sizes in Section 6.4. In Section 6.5, we have studied an estimator T of the form $a\bar{X} + bX_{(n)}$. In Section 6.6, the distribution of T is obtained.

For case (ii), Cohen (1955) and Deemer and Votaw (1955), have considered the maximum likelihood estimation of scale parameter σ. They have shown that mle of σ can even be infinite. Bain and Weeks (1964) have obtained the distribution of $S = \Sigma X_{(i)}$

However, it has a very complicated form. Bain *et al*. (1977) gave a beta approximation to the distribution of S/nx_o. For this case, only mle has been studied for estimation purposes so far. In last two sections, procedures for obtaining the best estimator of σ in the presence of a single outlier among all the estimators considered are suggested for both the cases.

CHAPTER 2

RECURRENCE RELATIONS AMONG THE MOMENTS OF ORDER STATISTICS FROM EXPONENTIAL AND TRUNCATED EXPONENTIAL DISTRIBUTIONS IN A SINGLE OUTLIER EXCHANGEABLE MODEL

2.1 Introduction

Suppose we have n independent random variables $X_1, \ldots, X_n$ such that (n-1) of them come from a distribution with pdf f(x) and cdf F(x), while the remaining one has the pdf g(x) and cdf G(x). This one random variable from g(x) corresponds to an outlying population. The case g(x) = f(x), represents the case of no outlier, and is called the homogeneous case. Let $X_{1:n} \leq \ldots \leq X_{n:n}$ be the order statistics obtained from $X_1, \ldots, X_n$. We denote $E(X_{r:n})$ by $\mu_{r:n}$ and the kth moment $E(X_{r:n}^k)$ by $\mu_{r:n}^{(k)}$, while the kth moment of rth order statistic in a random sample of size n from F(x) will be denoted by $\nu_{r:n}^{(k)}$ with $\nu_{r:n}^{(1)}$ by $\nu_{r:n}$ only. David and Shu (1978) have shown that the pdf of $X_{r:n}$ is given by

$$h_{r:n}(x) = \frac{(n-1)!}{(r-2)!\,(n-r)!} F^{r-2}(x)(1-F(x))^{n-r} f(x)\, G(x) + \frac{(n-1)!}{(r-1)!\,(n-r)!} F^{r-1}(x)(1-F(x))^{n-r} g(x) + \frac{(n-1)!}{(r-1)!(n-r-1)!} F^{r-1}(x)(1-F(x))^{n-r-1}(1-G(x))\, f(x), \tag{2.1.1}$$

where first term drops out if r = 1 and last term drops out if r = n.

The cdf of $X_{r:n}$ is simply given by

$$H_{r:n}(x) = F_{r:n-1}(x) + \binom{n-1}{r-1} F^{r-1}(x)(1-F(x))^{n-r} G(x),$$

$$r = 1,\dots,n-1,$$

$$H_{n:n}(x) = F^{n-1}(x)\, G(x), \tag{2.1.2}$$

where $F_{r:n-1}(x) = \sum_{i=r}^{n-1} \binom{n-1}{i} F^{i}(x)(1-F(x))^{n-i-1}$ is the cdf of rth order statistic in a sample of size (n-1) in homogeneous case.

David and Shu (1978) have also obtained the joint density of $X_{r:n}$ and $X_{s:n}$ $(1 \le r < s \le n)$ as

$$h_{rs:n}(x,y) = f_{r-1,s-1:n-1}(x,y)\, G(x) + f_{rs:n-1}(x,y)(1-G(y)) +$$

$$\frac{(n-1)!}{(r-1)!(n-s)!(s-r-1)!} \cdot F^{r-1}(x)$$

$$\cdot (F(y)-F(x))^{s-r-1}(1-F(y))^{n-s}[f(x)\, g(y) + g(x)\, f(y) +$$

$$(s-r-1)\left(\frac{G(y)-G(x)}{F(y)-F(x)}\right) f(x)\, f(y)] \quad \text{for} \quad x < y$$

$$= 0 \quad \text{elsewhere.} \tag{2.1.3}$$

Here for r = 1, first term drops out, for r = n second term drops out and for s = r+1, last term drops out, and

$$f_{rs:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F^{r-1}(x)(F(y)-F(x))^{s-r-1}$$

$$\cdot (1-F(y))^{n-s} f(x)\, f(y) \quad (x < y)$$

$$= 0 \quad \text{otherwise,}$$

is the joint density of rth and sth order statistic in a sample of size n in the homogeneous case.

For non-negative random variables, we can write (for example, see Rao, 1974, p. 94)

$$\mu_{r:n}^{(k)} = E(X_{r:n}^{k}) = \int_{0}^{\infty} x^{k}\, h_{r:n}(x)\, dx \tag{2.1.4}$$

$$= \int_{0}^{\infty} k\, x^{k-1}[1-H_{r:n}(x)]\, dx\, , \tag{2.1.5}$$

and

$$\nu_{r:n}^{(k)} = \int_{0}^{\infty} x^{k}\, f_{r:n}(x)\, dx \tag{2.1.6}$$

$$= \int_{0}^{\infty} k\, x^{k-1}[1-F_{r:n}(x)]\, dx\, . \tag{2.1.7}$$

In this case, some relations for single and product moments are given by David and Shu (1978), Balakrishnan (1988) and Balakrishnan and Ambagaspitiya (1988).

For $1 \leq r < s \leq n$, $\mu_{r,s:n} = E(X_{r:n}\, X_{s:n})$ is given by

$$\mu_{r,s:n} = \iint_{x<y} xy\, h_{rs:n}(x,y)\, dx\, dy\, . \tag{2.1.8}$$

In the homogeneous case, this product moment is denoted by $\nu_{r,s:n}$ and is given by

$$\nu_{r,s:n} = \iint_{x<y} xy\, f_{rs:n}(x,y)\, dx\, dy\, . \tag{2.1.9}$$

In this chapter, we derive some specific recurrence relations for single and product moments of the truncated exponential model having a pdf

$$f(x) = \begin{cases} \dfrac{1}{1-e^{-x_o}}\, e^{-x} & 0 < x \leq x_o \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} \dfrac{\alpha}{1-e^{-\alpha x_o}}\, e^{-\alpha x} & 0 < x \leq x_o \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and the truncation point x_0 is fixed and assumed to be known. The results derived here for single moments are generalization of similar results for the homogeneous case obtained by Joshi (1978), which are given by

$$\nu_{r:n}^{(k)} = \frac{1}{F_0}\,\nu_{r-1:n-1}^{(k)} + \frac{k}{n}\,\nu_{r:n}^{(k-1)} - \frac{e^{-x_0}}{F_0}\,\nu_{r:n-1}^{(k)}, \quad 1 \le r \le n-1,$$

$$\nu_{n:n}^{(k)} = \frac{1}{F_0}\,\nu_{n-1:n-1}^{(k)} + \frac{k}{n}\,\nu_{n:n}^{(k-1)} - \frac{x_0^k\, e^{-x_0}}{F_0}, \quad F_0 = 1-e^{-x_0}. \qquad (2.1.10)$$

The moments $\mu_{r:n}^{(k)}$ and $\mu_{r,s:n}$ will be used for studying the robustness properties of some estimators of scale parameter under this set up. Note that for $x_0 = \infty$, there is no truncation and the results are for the exponential distribution with a single outlier. This case has been widely studied by Kale and Sinha (1971), Joshi (1972), Chikkagoudar and Kunchur (1980), and Gross _et al_. (1986).

2.2 Recurrence relations for single moments

The following theorem gives the main result of this section, where a recurrence relation linking $\mu_{r:n}^{(k)}$ with lower order moments is derived. This allows us to evaluate all the moments in a systematic manner.

THEOREM 2.2.1: For $k = 1,2,\ldots,$

$$\mu_{r:n}^{(k)} = \frac{1}{(n-1+\alpha)}\Big[\frac{(n-1)}{F_0}\big(\mu_{r-1:n-1}^{(k)} - e^{-x_0}\,\mu_{r:n-1}^{(k)}\big) + \frac{\alpha}{G_0}\big(\nu_{r-1:n-1}^{(k)} - e^{-\alpha x_0}\,\nu_{r:n-1}^{(k)}\big) + k\,\mu_{r:n}^{(k-1)}\Big],$$

$$\text{for } r = 1,2,\ldots,n-1, \qquad (2.2.1)$$

and

$$\mu_{n:n}^{(k)} = \frac{1}{n-1+\alpha}\left[\frac{(n-1)}{F_o}\left(\mu_{n-1:n-1}^{(k)} - x_o^k e^{-x_o}\right) + \frac{\alpha}{G_o}\left(\nu_{n-1:n-1}^{(k)} - x_o^k e^{-\alpha x_o}\right) + k\,\mu_{n:n}^{(k-1)}\right], \qquad (2.2.2)$$

where $F_o = 1 - e^{-x_o}$,

$G_o = 1 - e^{-\alpha x_o}$,

$\mu_{r:n}^{(0)} = 1, \quad 1 \le r \le n,$

$\mu_{0:t}^{(k)} = 0, \quad k = 1,2,\ldots, \quad t = 0,1,2,\ldots,$

$\nu_{0:t}^{(k)} = 0, \quad k = 1,2,\ldots, \quad t = 0,1,2,\ldots.$

Proof: From equations (2.1.2) and (2.1.5), we have

$$\mu_{r:n}^{(k)} = \int_0^\infty k\,x^{k-1}\left[1 - F_{r:n-1}(x) - \binom{n-1}{r-1} F^{r-1}(x)(1-F(x))^{n-r}\, G(x)\right]dx.$$

For truncated exponential model, it reduces to

$$\mu_{r:n}^{(k)} = \int_0^{x_o} k\,x^{k-1}\left\{1 - \sum_{i=r}^{n-1}\binom{n-1}{i}\left(\frac{1-e^{-x}}{F_o}\right)^{i}\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-1-i}\right\}dx$$
$$- \binom{n-1}{r-1}\int_0^{x_o} k\,x^{k-1}\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-r}\left(\frac{1-e^{-\alpha x}}{G_o}\right)dx.$$

Using equation (2.1.7), we obtain

$$\mu_{r:n}^{(k)} = \nu_{r:n-1}^{(k)} - I_1 + I_2, \qquad (2.2.3)$$

where $I_1 = \binom{n-1}{r-1}\int_0^{x_o} k\,x^{k-1}\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-r}\frac{1}{G_o}\,dx$

and $I_2 = \binom{n-1}{r-1}\frac{1}{G_o}\int_0^{x_o} k\,x^{k-1}\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-r} e^{-\alpha x}\,dx.$

Integrating the integral in I_1 by treating x^{k-1} for integration and $(\frac{1-e^{-x}}{F_o})^{r-1}(1-\frac{1-e^{-x}}{F_o})^{n-r}$ for differentiation, we get

$$I_1 = \binom{n-1}{r-1}\frac{1}{G_o}\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\left(1-\frac{1-e^{-x}}{F_o}\right)^{n-r} x^k \Big|_o^{x_o} -$$

$$\frac{1}{G_o}\binom{n-1}{r-1}\int_o^{x_o} x^k\Big[(r-1)\left(\frac{1-e^{-x}}{F_o}\right)^{r-2}\left(1-\frac{1-e^{-x}}{F_o}\right)^{n-r}\frac{e^{-x}}{F_o} -$$

$$(n-r)\left(1-\frac{1-e^{-x}}{F_o}\right)^{n-r-1}\frac{e^{-x}}{F_o}\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\Big]\,dx.$$

On using equation (2.1.6), it gives

$$I_1 = -\frac{1}{G_o}\left(\nu^{(k)}_{r-1:n-1} - \nu^{(k)}_{r:n-1}\right).$$

Substituting this value of I_1 in equation (2.2.3), we obtain

$$\mu^{(k)}_{r:n} = \nu^{(k)}_{r:n-1} + \frac{1}{G_o}\left(\nu^{(k)}_{r-1:n-1} - \nu^{(k)}_{r:n-1}\right) + I_2 . \tag{2.2.4}$$

Integrating the original integral in I_2 by treating $e^{-\alpha x}$ for integration, and $(\frac{1-e^{-x}}{F_o})^{r-1}(1-\frac{1-e^{-x}}{F_o})^{n-r}x^{k-1}$ for differentiation we have

$$G_o\,I_2 = \binom{n-1}{r-1}\Big[\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\left(1-\frac{1-e^{-x}}{F_o}\right)^{n-r}\frac{e^{-\alpha x}}{-\alpha}\,k\,x^{k-1}\Big|_o^{x_o} +$$

$$\int_o^{x_o} k(k-1)x^{k-2}\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\left(1-\frac{1-e^{-x}}{F_o}\right)^{n-r}\frac{e^{-\alpha x}}{\alpha}\,dx +$$

$$\int_o^{x_o} k\,x^{k-1}(r-1)\left(\frac{1-e^{-x}}{F_o}\right)^{r-2}\left(1-\frac{1-e^{-x}}{F_o}\right)^{n-r}\frac{e^{-x}}{F_o}\frac{e^{-\alpha x}}{\alpha}\,dx\Big] - \frac{n-r}{\alpha}\,I, \tag{2.2.5}$$

where $I = \binom{n-1}{r-1}\int_o^{x_o} k\,x^{k-1}\left(\frac{1-e^{-x}}{F_o}\right)^{r-1}\left(1-\frac{1-e^{-x}}{F_o}\right)^{n-r-1} e^{-\alpha x}\,\frac{e^{-x}}{F_o}\,dx.$

For simplifying I_2, consider the decomposition of $\{1-\frac{1-e^{-x}}{F_o}\}^{n-r}$

as $\{1 - \frac{1-\bar{e}^{x}}{F_o}\}^{n-r-1} \{1 - \frac{1}{F_o} + \frac{\bar{e}^{x}}{F_o}\}$. This gives

$$G_o I_2 = \binom{n-1}{r-1} \int_0^{x_o} k\, x^{k-1} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r-1} \bar{e}^{\alpha x} \left(1 - \frac{1}{F_o}\right) dx + \tag{2.2.6}$$

Comparing equations (2.2.5) and (2.2.6), we get

$$\left(\frac{n-r}{\alpha} + 1\right) I = \binom{n-1}{r-1} \Big[\int_0^{x_o} k(k-1) x^{k-2} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} \frac{\bar{e}^{\alpha x}}{\alpha}\, dx +$$

$$\int_0^{x_o} k\, x^{k-1} (r-1) \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-2} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} \frac{\bar{e}^{x}\bar{e}^{\alpha x}}{\alpha F_o}\, dx -$$

$$\int_0^{x_o} k\, x^{k-1} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r-1} \bar{e}^{\alpha x} \left(1 - \frac{1}{F_o}\right) dx \Big].$$

Substituting the value of I in equation (2.2.5), it reduces to

$$G_o\, I_2 = \binom{n-1}{r-1} \Big[\frac{k}{\alpha} \int_0^{x_o} (k-1) x^{k-2}\, \bar{e}^{\alpha x} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} dx +$$

$$\frac{r-1}{\alpha} \int_0^{x_o} k\, x^{k-1} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-2} \frac{\bar{e}^{x}}{F_o} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} \bar{e}^{\alpha x}\, dx -$$

$$\frac{(n-r)}{n-r+\alpha} \frac{k}{\alpha} \int_0^{x_o} (k-1) x^{k-2} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} \bar{e}^{\alpha x}\, dx -$$

$$\frac{(n-r)}{(n-r+\alpha)} \frac{r-1}{\alpha} \int_0^{x_o} k\, x^{k-1} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-2} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} \frac{\bar{e}^{x}\bar{e}^{\alpha x}}{F_o}\, dx +$$

$$\frac{(n-r)}{(n-r+\alpha)} \left(1 - \frac{1}{F_o}\right) \int_0^{x_o} k\, x^{k-1} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r-1} \bar{e}^{\alpha x}\, dx \Big],$$

which simplifies to

$$G_o\, I_2 = \binom{n-1}{r-1} \Big[\frac{k}{(n-r+\alpha)} \int_0^{x_o} (k-1) x^{k-2} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} \bar{e}^{\alpha x}\, dx +$$

$$\frac{(r-1)}{(n-r+\alpha)} \int_0^{x_o} k\, x^{k-1} \left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-2} \left(1 - \frac{1-\bar{e}^{x}}{F_o}\right)^{n-r} \frac{\bar{e}^{x}\bar{e}^{\alpha x}}{F_o}\, dx +$$

$$\frac{(n-r)}{(n-r+\alpha)}\left(1-\frac{1}{F_o}\right)\int_o^{x_o} k\,x^{k-1}\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1}\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-r-1}\bar{e}^{\alpha x}\,d$$

Writing $\bar{e}^{x}$ as $[1-(1-\bar{e}^{x})]$ in the second integral on r.h.s., and then considering the two integrals separately, we have

$$G_oI_2 = \binom{n-1}{r-1}\frac{k}{(n-r+\alpha)}\int_o^{x_o}(k-1)x^{k-2}\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1}\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-r}\bar{e}^{\alpha x}\,dx +$$

$$\binom{n-1}{r-1}\frac{(r-1)}{(n-r+\alpha)}\int_o^{x_o}k\,x^{k-1}\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-2}\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-r}\frac{\bar{e}^{\alpha x}}{F_o}dx -$$

$$G_o\frac{(r-1)}{(n-r+\alpha)}I_2 + \frac{(n-r)}{(n-r+\alpha)}\left(1-\frac{1}{F_o}\right)\binom{n-1}{r-1}\int_o^{x_o}k\,x^{k-1}\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1}$$

$$\cdot\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-r-1}\bar{e}^{\alpha x}\,dx,$$

which gives

$$(n-1+\alpha)G_oI_2 = \binom{n-1}{r-1}\Big[k\int_o^{x_o}(k-1)x^{k-2}\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1}\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-r}\bar{e}^{\alpha x}\,dx +$$

$$(r-1)\int_o^{x_o}k\,x^{k-1}\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-2}\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-r}\frac{\bar{e}^{\alpha x}}{F_o}\,dx +$$

$$(n-r)\left(1-\frac{1}{F_o}\right)\int_o^{x_o}k\,x^{k-1}\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1}\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-r-1}$$

$$\cdot\,\bar{e}^{\alpha x}\,dx\Big].$$

Making use of equation (2.2.4), it reduces to

$$(n-1+\alpha)I_2 = k\left(\mu_{r:n}^{(k-1)} - \nu_{r:n-1}^{(k-1)}\left(1-\frac{1}{G_o}\right) - \frac{1}{G_o}\nu_{r-1:n-1}^{(k-1)}\right) +$$

$$\frac{(n-1)}{F_o}\left(\mu_{r-1:n-1}^{(k)} - \nu_{r-1:n-2}^{(k)}\left(1-\frac{1}{G_o}\right) - \frac{1}{G_o}\nu_{r-2:n-2}^{(k)}\right) +$$

$$(n-1)\left(1-\frac{1}{F_o}\right)\left(\mu_{r:n-1}^{(k)} - \nu_{r:n-2}^{(k)}\left(1-\frac{1}{G_o}\right) - \frac{\nu_{r-1:n-2}^{(k)}}{G_o}\right).$$

Using equation (2.1.10), we get

$$(n-1+\alpha)I_2 = k[\mu_{r:n}^{(k-1)} - \nu_{r:n-1}^{(k-1)}(1-\frac{1}{G_o}) - \frac{1}{G_o}\nu_{r-1:n-1}^{(k-1)}] +$$

$$(n-1)[\frac{\mu_{r-1:n-1}^{(k)}}{F_o} + (1-\frac{1}{F_o})\mu_{r:n-1}^{(k)} -$$

$$(1-\frac{1}{G_o})(\nu_{r:n-1}^{(k)} - \frac{k}{(n-1)}\nu_{r:n-1}^{(k-1)}) -$$

$$\frac{1}{G_o}(\nu_{r-1:n-1}^{(k)} - \frac{k}{(n-1)}\nu_{r-1:n-1}^{(k-1)})].$$

On simplification it reduces to

$$(n-1+\alpha)I_2 = k\,\mu_{r:n}^{(k-1)} + \frac{(n-1)}{F_o}\mu_{r-1:n-1}^{(k)} + (n-1)(1-\frac{1}{F_o})\mu_{r:n-1}^{(k)} -$$

$$(n-1)(1-\frac{1}{G_o})\nu_{r:n-1}^{(k)} - (n-1)\frac{1}{G_o}\nu_{r-1:n-1}^{(k)}.$$

Substituting this value of I_2 in equation (2.2.4), we obtain

$$\mu_{r:n}^{(k)} = (1-\frac{1}{G_o})\frac{\alpha}{(n-1+\alpha)}\nu_{r:n-1}^{(k)} + \frac{k}{(n-1+\alpha)}\mu_{r:n}^{(k-1)} +$$

$$\frac{1}{G_o}\frac{\alpha}{(n-1+\alpha)}\nu_{r-1:n-1}^{(k)} + \frac{(n-1)}{(n-1+\alpha)}\frac{1}{F_o}\mu_{r-1:n-1}^{(k)} +$$

$$\frac{(n-1)}{(n-1+\alpha)}(1-\frac{1}{F_o})\mu_{r:n-1}^{(k)}.$$

After some simplification, we get equation (2.2.1).

The proof of equation (2.2.2) is analogous. Now on using equation (2.1.2) and equation (2.1.5), we have

$$\mu_{n:n}^{(k)} = \int_0^{x_o} k\,x^{k-1}(1 - \frac{(1-e^{-x})^{n-1}(1-e^{-\alpha x})}{F_o^{n-1}G_o})\,dx,$$

which gives

$$\mu_{n:n}^{(k)} = x_o^k - I_1 + I_2, \tag{2.2.7}$$

where

$$I_1 = \int_0^{x_0} k\, x^{k-1} \frac{(1-e^{-x})^{n-1}}{F_0^{n-1} G_0}\, dx,$$

$$I_2 = \int_0^{x_0} k\, x^{k-1} \frac{(1-e^{-x})^{n-1}\, e^{-\alpha x}}{F_0^{n-1} G_0}\, dx.$$

Using equation (2.1.7) for the truncated exponential model with $r = n-1$ and n replaced by $n-1$, it gives

$$\nu_{n-1:n-1}^{(k)} = \int_0^{x_0} k\, x^{k-1} \left(1 - \frac{(1-e^{-x})^{n-1}}{F_0^{n-1}}\right) dx.$$

Hence

$$G_0 I_1 = x_0^k - \nu_{n-1:n-1}^{(k)}.$$

For simplifying I_2, consider the decomposition of $(1-e^{-x})^{n-1}$ as $(1-e^{-x})^{n-2}(1-e^{-x})$. It yields

$$I_2 = \int_0^{x_0} k\, x^{k-1} \frac{(1-e^{-x})^{n-2} e^{-\alpha x}}{F_0^{n-1} G_0}\, dx - \int_0^{x_0} k\, x^{k-1} \frac{(1-e^{-x})^{n-2} e^{-x(\alpha+1)}}{F_0^{n-1} G_0}\, dx.$$

(2.2.8)

But I_2 can also be written as

$$F_0^{n-1} G_0 I_2 = -k\, x^{k-1} (1-e^{-x})^{n-1} \frac{e^{-\alpha x}}{\alpha}\Big|_0^{x_0} + (n-1) \int_0^{x_0} k\, x^{k-1} (1-e^{-x})^{n-2}$$

$$\cdot \frac{e^{-x(\alpha+1)}}{\alpha}\, dx + k \int_0^{x_0} (k-1) x^{k-2} (1-e^{-x})^{n-1} \frac{e^{-\alpha x}}{\alpha}\, dx.$$

(2.2.9)

On comparing equations (2.2.8) and (2.2.9), we get

$$\left(\frac{n-1}{\alpha} + 1\right) I = \frac{k}{\alpha} x_0^{k-1} e^{-\alpha x_0} (1-e^{-x_0})^{n-1} + \int_0^{x_0} k\, x^{k-1} (1-e^{-x})^{n-2} e^{-\alpha x}\, dx$$

$$- \int_0^{x_0} k(k-1)\, x^{k-2} (1-e^{-x})^{n-1} \frac{e^{-\alpha x}}{\alpha}\, dx,$$

where $I = \int_o^{x_o} k\, x^{k-1}(1-e^{-x})^{n-2}\, e^{-x(\alpha+1)}\, dx.$

Substituting this value of I in equation (2.2.9), we get

$$F_o^{n-1}G_oI_2 = \frac{-k\, x_o^{k-1}\, e^{-\alpha x_o}(1-e^{-x_o})^{n-1}}{(n-1+\alpha)} + \frac{(n-1)}{\alpha}\left[\frac{\alpha}{(n-1+\alpha)} \int_o^{x_o} k\, x^{k-1}\right.$$

$$.(1-e^{-x})^{n-2}\, e^{-\alpha x}\, dx - \frac{k}{(n-1+\alpha)} \int_o^{x_o} (k-1)x^{k-2}(1-e^{-x})^{n-1}e^{-\alpha x}\, dx$$

$$+ k \int_o^{x_o} (k-1)x^{k-2}(1-e^{-x})^{n-1}\, \frac{e^{-\alpha x}}{\alpha}\, dx,$$

which can be simplified to

$$F_o^{n-1}G_oI_2 = \frac{-k\, x_o^{k-1}\, e^{-\alpha x_o}\, (1-e^{-x_o})^{n-1}}{(n-1+\alpha)} + I_3 + I_4, \tag{2.2.10}$$

where $I_3 = \frac{(n-1)}{(n-1+\alpha)} \int_o^{x_o} k\, x^{k-1}(1-e^{-x})^{n-2}\, e^{-\alpha x}\, dx,$

$$I_4 = \left[1 - \frac{(n-1)}{(n-1+\alpha)}\right] \frac{k}{\alpha} \int_o^{x_o} (k-1)x^{k-2}(1-e^{-x})^{n-1}\, e^{-\alpha x}\, dx.$$

Using equation (2.1.5) for sample size (n-1) and r = n-1, we get

$$\mu_{n-1:n-1}^{(k)} = x_o^k - \int_o^{x_o} k\, \frac{x^{k-1}(1-e^{-x})^{n-2}}{F_o^{n-2}G_o}\, dx +$$

$$\int_o^{x_o} k\, x^{k-1}\, \frac{(1-e^{-x})^{n-2}\, e^{-\alpha x}}{F_o^{n-2}G_o}\, dx,$$

which gives

$$(F_o^{n-2}G_o)^{-1}\, \frac{(n-1+\alpha)}{(n-1)}\, I_3 = \mu_{n-1:n-1}^{(k)} - x_o^k + \int_o^{x_o} k\, x^{k-1}\, \frac{(1-e^{-x})^{n-2}}{F_o^{n-2}G_o}\, dx.$$

Using equation (2.1.7) for sample size (n-2), it reduces to

$$\frac{(n-1+\alpha)}{(n-1)}\, \frac{1}{F_o^{n-2}G_o}\, I_3 = \mu_{n-1:n-1}^{(k)} - x_o^k + \frac{x_o^k - \nu_{n-2:n-2}^{(k)}}{G_o}\, .$$

Similarly using equation (2.1.5) and (2.1.7), we obtain the value of I_4 as

$$\frac{(n-1+\alpha)}{\alpha} \times \frac{1}{F_o^{n-1} G_o} I_4 = \frac{k}{\alpha} (\mu_{n:n}^{(k-1)} - x_o^{k-1} + \frac{x_o^{k-1} - \nu_{n-1:n-1}^{(k-1)}}{G_o}).$$

Substituting these values of I_3 and I_4 in equation (2.2.10), we have

$$I_2 = \frac{-k\, x_o^{k-1} e^{-\alpha x_o}}{(n-1+\alpha) G_o} + \frac{(n-1)}{(n-1+\alpha)} \cdot \frac{1}{F_o} (\mu_{n-1:n-1}^{(k)} - x_o^k + \frac{x_o^k - \nu_{n-2:n-2}^{(k)}}{G_o})$$
$$+ \frac{\alpha}{(n-1+\alpha)} \frac{k}{\alpha} (\mu_{n:n}^{(k-1)} - x_o^{k-1} + \frac{x_o^{k-1} - \nu_{n-1:n-1}^{(k-1)}}{G_o}).$$

Substituting the values of I_1 and I_2 in equation (2.2.7), it gives

$$\mu_{n:n}^{(k)} = x_o^k - \frac{x_o^k - \nu_{n-1:n-1}^{(k)}}{G_o} + \frac{(n-1)}{(n-1+\alpha)} \frac{1}{F_o} (\mu_{n-1:n-1}^{(k)} - x_o^k +$$
$$\frac{x_o^k - \nu_{n-2:n-2}^{(k)}}{G_o}) - \frac{k\, x_o^{k-1} e^{-\alpha x_o}}{(n-1+\alpha) G_o} + \frac{\alpha}{(n-1+\alpha)} \frac{k}{\alpha} (\mu_{n:n}^{(k-1)} - x_o^{k-1} +$$
$$\frac{x_o^{k-1} - \nu_{n-1:n-1}^{(k-1)}}{G_o}).$$

Using equation (2.1.10) for $\nu_{n-2:n-2}^{(k)}$, and simplifying, it reduces to

$$\mu_{n:n}^{(k)} = \frac{1}{(n-1+\alpha)} [\frac{(n-1)}{F_o} (\mu_{n-1:n-1}^{(k)} - x_o^k e^{-x_o}) + \frac{\alpha}{G_o} (\nu_{n-1:n-1}^{(k)} - x_o^k e^{-\alpha x_o})$$
$$+ k\, \mu_{n:n}^{(k-1)}].$$

This completes the proof of the theorem. It can be seen that on taking $\alpha = 1$ in equations (2.2.1) and (2.2.2), we immediately get the recurrence relations given at equation (2.1.10).

COROLLARY 2.2.1: For samples containing a single outlier from an exponential distribution

$$\mu_{r:n}^{(k)} = \frac{1}{(n-1+\alpha)}\left[(n-1)\ \mu_{r-1:n-1}^{(k)} + \alpha\ \nu_{r-1:n-1}^{(k)} + k\ \mu_{r:n}^{(k-1)}\right]. \tag{2.2.11}$$

Proof: The proof follows immediately on taking limit as $x_o \to \infty$ in equation (2.2.1) and equation (2.2.2).

In this case, the moments $\mu_{r:n}^{(k)}$ can be obtained by using equation (2.2.11). However, means, variances and covariances can be more easily calculated by the method described in Joshi (1972).

2.3 Recurrence relations for product moments

It is also possible to derive recurrence relations for some product moments. Here we first obtain such relations for $\mu_{1,2:n}$ and $\mu_{n-1,n:n}$. For general $\mu_{r,s:n}$ the integrals involved are very complicated. These relations also extend the recurrence relations obtained for $\mu_{r,s:n}$ for the untruncated case by Khan et al. (1983) and Balakrishnan and Joshi (1984).

THEOREM 2.3.1: For $n = 3,4,\ldots,$

$$\begin{aligned}\mu_{1,2:n} = \frac{1}{(n-2+\alpha)}\Big[&\mu_{1:n} + \frac{(n-1)(\alpha+1)}{2\alpha}\mu_{1:n}^{(2)} + \frac{e^{-x_o}}{F_o}\frac{(n-1)(\alpha+1)}{2\alpha}\mu_{1:n-1}^{(2)} \\ &- \frac{e^{-x_o}}{F_o}(n-1)\mu_{1,2:n-1} - \frac{\alpha e^{-\alpha x_o}}{G_o}\nu_{1,2:n-1} - \frac{(1-\alpha)}{\alpha G_o}\nu_{1:n-1} \\ &+ \frac{e^{-\alpha x_o}}{G_o}\left(1 + \frac{(n-1)(1-\alpha)}{2\alpha}\right)\nu_{1:n-1}^{(2)} + \frac{(n-1)(1-\alpha)}{2\alpha}\frac{e^{-x_o}}{F_o}\cdot \\ &\cdot\frac{e^{-\alpha x_o}}{G_o}\nu_{1:n-2}^{(2)}\Big].\end{aligned} \tag{2.3.1}$$

For $n \geq 2$,

$$\mu_{n-1,n:n} = \mu_{n-1:n} - (n-1)\frac{x_o e^{-x_o}}{F_o}\mu_{n-1:n-1} + \frac{(n-1)\mu^{(2)}_{n-1:n-1}}{F_o}\frac{(\alpha+1)}{\alpha}$$
$$- \frac{(n-1)(\alpha+1)}{2\alpha}\mu^{(2)}_{n:n} - \frac{\nu_{n-1:n-1}}{G_o}[e^{-\alpha x_o}(\frac{1}{\alpha} - 1 + x_o) - \frac{(1-\alpha)}{\alpha}$$
$$+ \frac{\nu^{(2)}_{n-1:n-1}}{G_o} - \frac{(1-\alpha)(n-1)\ e^{-x_o}\ x_o^2}{2\alpha\ F_o}, \qquad (2.3.2)$$

where the notations are same as in Theorem 2.2.1.

Proof: The method used in proving these results is analogous to the one given by Joshi (1982). We write

$$\mu_{1:n} = E(X_{1:n}\ X^o_{2:n}).$$

From equation (2.1.3) and equation (2.1.8), we have

$$\mu_{1:n} = (n-1)(n-2)\iint_w x(1 - \frac{1-e^{-y}}{F_o})^{n-3}(1 - \frac{1-e^{-\alpha y}}{G_o})\frac{e^{-x}}{F_o}\frac{e^{-y}}{F_o}\,dx\,dy$$
$$+ (n-1)[\iint_w x(1 - \frac{1-e^{-y}}{F_o})^{n-2}\frac{e^{-x}}{F_o}\frac{\alpha e^{-\alpha y}}{G_o}\,dx\,dy$$
$$+ \iint_w x(1 - \frac{1-e^{-y}}{F_o})^{n-2}\frac{e^{-y}}{F_o}\frac{\alpha e^{-\alpha x}}{G_o}\,dx\,dy],$$

where $w = \{(x,y): 0 < x < y \leq x_o\}$ is the region of integration. For simplifying, we write $\mu_{1:n}$ as

$$\mu_{1:n} = I_1 + I_2 + I_3. \qquad (2.3.3)$$

Then we consider these integrals separately. Now

$$I_1 = (n-1)(n-2)\int_o^{x_o} x\frac{e^{-x}}{F_o}J_1\,dx,$$

where

$$J_1 = \int_x^{x_0} \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-3} \left(1 - \frac{1-e^{-\alpha y}}{G_0}\right) \frac{e^{-y}}{F_0}\, dy,$$

which follows on integration by parts by treating 1 for integration and $(1 - \frac{1-e^{-y}}{F_0})^{n-3} (1 - \frac{1-e^{-\alpha y}}{G_0}) \frac{e^{-y}}{F_0}$ for differentiation. Substituting this value in I_1, we get

$$I_1 = (n-1)(n-2)\Big[- \int_0^{x_0} x^2 \left(1 - \frac{1-e^{-x}}{F_0}\right)^{n-2} \left(1 - \frac{1-e^{-\alpha x}}{G_0}\right) \frac{e^{-x}}{F_0}\, dx +$$

$$(n-2) \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-3} \left(1 - \frac{1-e^{-\alpha y}}{G_0}\right) \frac{e^{-y}}{F_0} \frac{e^{-x}}{F_0}\, dx\, dy +$$

$$\iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-2} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha y}}{G_0}\, dx\, dy - \frac{e^{-x_0}}{F_0} \int_0^{x_0} x^2 \left(1 - \frac{1-e^{-x}}{F_0}\right)^{n-3}$$

$$\cdot \left(1 - \frac{1-e^{-\alpha x}}{G_0}\right) \frac{e^{-x}}{F_0}\, dx + \frac{e^{-x_0}}{F_0} (n-3) \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-4} \left(1 - \frac{1-e^{-\alpha y}}{G_0}\right)$$

$$\cdot \frac{e^{-y} e^{-x}}{F_0^2}\, dx\, dy + \frac{e^{-x_0}}{F_0} \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-3} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha y}}{G_0}\, dx\, dy\Big].$$

Simplifying for I_2 and I_3 in an identical manner, we get

$$I_2 = (n-1)\alpha\Big[- \int_0^{x_0} x^2 \left(1 - \frac{1-e^{-x}}{F_0}\right)^{n-2} \frac{e^{-\alpha x}}{G_0} \frac{e^{-x}}{F_0}\, dx +$$

$$(n-2) \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-3} \frac{e^{-y}}{F_0} \frac{e^{-\alpha y}}{G_0} \frac{e^{-x}}{F_0}\, dx\, dy$$

$$+ \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-2} \frac{\alpha e^{-\alpha y}}{G_0} \frac{e^{-x}}{F_0}\, dx\, dy\Big],$$

$$I_3 = (n-1)\Big[- \int_0^{x_0} x^2 \left(1 - \frac{1-e^{-x}}{F_0}\right)^{n-2} \frac{e^{-x}}{F_0} \frac{\alpha e^{-\alpha x}}{G_0}\, dx +$$

$$(n-2) \iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-3} \left(\frac{e^{-y}}{F_0}\right)^2 \frac{\alpha e^{-\alpha x}}{G_0}\, dx\, dy +$$

$$\iint_W xy \left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-2} \frac{e^{-y}}{F_0} \frac{\alpha e^{-\alpha x}}{G_0}\, dx\, dy\Big].$$

Substituting the values of I_1, I_2, I_3 in equation (2.3.3), and using equations (2.1.5) and (2.1.8), it gives

$$\mu_{1:n} = -(n-1)\mu_{1:n}^{(2)} - \frac{e^{-x_o}}{F_o}(n-1)\mu_{1:n-1}^{(2)} + (n-1)\int_o^{x_o} x^2\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-2}$$

$$\cdot\left(1 - \frac{1-e^{-\alpha x}}{G_o}\right)\frac{e^{-x}}{F_o}\,dx - (n-1)\int_o^{x_o} x^2\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-2}\frac{\alpha e^{-\alpha x}}{G_o}\frac{e^{-x}}{F_o}\,dx$$

$$+ (n-2)\mu_{1,2:n} + \frac{e^{-x_o}}{F_o}(n-1)\mu_{1,2:n-1} + I_5, \tag{2.3.4}$$

where

$$I_5 = (n-1)(n-2)\Big[\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-2}\frac{e^{-x}}{F_o}\frac{\alpha e^{-\alpha y}}{G_o}\,dx\,dy +$$

$$\frac{e^{-x_o}}{F_o}\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-3}\frac{e^{-x}}{F_o}\frac{\alpha e^{-\alpha y}}{G_o}\,dx\,dy\Big] +$$

$$(n-1)\alpha\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-2}\frac{\alpha e^{-\alpha y}}{G_o}\frac{e^{-x}}{F_o}\,dx\,dy +$$

$$(n-1)\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-2}\frac{e^{-y}}{F_o}\frac{\alpha e^{-\alpha x}}{G_o}\,dx\,dy.$$

Decomposing $\left\{1 - \frac{1-e^{-y}}{F_o}\right\}^{n-2}$ as $\left\{1 - \frac{1-e^{-y}}{F_o}\right\}^{n-3}\left\{1 - \frac{1}{F_o} + \frac{e^{-y}}{F_o}\right\}$ in first integral, we get

$$I_5 = (n-1)(n-2)\Big[\left(1 - \frac{1}{F_o}\right)\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-3}\frac{e^{-x}}{F_o}\frac{\alpha e^{-\alpha y}}{G_o}\,dx\,dy +$$

$$\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-3}\frac{e^{-x}}{F_o}\frac{\alpha e^{-\alpha y}}{G_o}\frac{e^{-y}}{F_o}\,dx\,dy +$$

$$\frac{e^{-x_o}}{F_o}\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-3}\frac{e^{-x}}{F_o}\frac{\alpha e^{-\alpha y}}{G_o}\,dx\,dy\Big] +$$

$$(n-1)\alpha\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-2}\frac{\alpha e^{-\alpha y}}{G_o}\frac{e^{-x}}{F_o}\,dx\,dy +$$

$$(n-1)\iint_W xy\left(1 - \frac{1-e^{-y}}{F_o}\right)^{n-2}\frac{e^{-y}}{F_o}\frac{\alpha e^{-\alpha x}}{G_o}\,dx\,dy.$$

Writing $\frac{\bar{e}^{\alpha y}}{G_o}$ as $(1 - \frac{1-\bar{e}^{\alpha y}}{G_o} + \frac{1}{G_o} - 1)$ in second integral and simplifying, it gives

$$I_5 = (n-1)(n-2)\alpha[\iint_W xy(1 - \frac{1-\bar{e}^y}{F_o})^{n-3} (1 - \frac{1-\bar{e}^{\alpha y}}{G_o}) \frac{\bar{e}^x}{F_o} \frac{\bar{e}^y}{F_o} dx\, dy +$$

$$\frac{e^{-\alpha x_o}}{G_o} \iint_W xy(1 - \frac{1-\bar{e}^y}{F_o})^{n-3} \frac{\bar{e}^x}{F_o} \frac{\bar{e}^y}{F_o} dx\, dy] +$$

$$(n-1)\alpha \iint_W xy(1 - \frac{1-\bar{e}^y}{F_o})^{n-2} \frac{\alpha\bar{e}^{\alpha y}}{G_o} \frac{\bar{e}^x}{F_o} dx\, dy +$$

$$(n-1) \iint_W xy(1 - \frac{1-\bar{e}^y}{F_o})^{n-2} \frac{\bar{e}^y}{F_o} \frac{\alpha\bar{e}^{\alpha x}}{G_o} dx\, dy.$$

Now using equations (2.1.8) and (2.1.9), we get

$$I_5 = \alpha\mu_{1,2:n} + \frac{\alpha e^{-\alpha x_o}}{G_o} \nu_{1,2:n-1} + (n-1)(1-\alpha) \int_o^{x_o} y(1 - \frac{1-\bar{e}^y}{F_o})^{n-2}$$

$$\cdot \frac{\bar{e}^y}{F_o} I_6\, dy,$$

where

$$I_6 = \int_o^y x \frac{\alpha\bar{e}^{\alpha x}}{G_o} dx$$

$$= - \frac{y\bar{e}^{\alpha y}}{G_o} - \frac{\bar{e}^{\alpha y}}{\alpha G_o} + \frac{1}{\alpha G_o} .$$

Substituting this value in I_5, we have

$$I_5 = \alpha\mu_{1,2:n} + \frac{\alpha e^{-\alpha x_o}}{G_o} \nu_{1,2:n-1} + (n-1)(1-\alpha)[- \int_o^{x_o} y^2(1 - \frac{1-\bar{e}^y}{F_o})^{n-2}$$

$$\cdot \frac{\bar{e}^y}{F_o} \frac{\bar{e}^{\alpha y}}{G_o} dy - \int_o^{x_o} y(1 - \frac{1-\bar{e}^y}{F_o})^{n-2} \frac{\bar{e}^y}{F_o} \frac{\bar{e}^{\alpha y}}{\alpha G_o} dy +$$

$$\int_o^{x_o} y(1 - \frac{1-\bar{e}^y}{F_o})^{n-2} \frac{\bar{e}^y}{F_o} \frac{1}{\alpha G_o} dy],$$

using equation (2.1.6) and making the transformation $x = y$, we obtain

$$I_5 = \alpha\mu_{1,2:n} + \frac{\alpha e^{-\alpha x_o}}{G_o}\nu_{1,2:n-1} + \frac{(1-\alpha)}{\alpha G_o}\nu_{1:n-1} + (n-1)(1-\alpha)$$

$$\cdot\left[-\int_o^{x_o} x^2\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-2}\frac{\bar{e}^{x}}{F_o}\frac{\bar{e}^{\alpha x}}{G_o}\,dx - \int_o^{x_o} x\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-2}\right.$$

$$\left.\cdot\frac{\bar{e}^{x}}{F_o}\frac{\bar{e}^{\alpha x}}{\alpha G_o}\,dx\right].$$

Substituting this value of I_5 in equation (2.3.4) and simplifying, it reduces to

$$\mu_{1:n} = -(n-1)\mu_{1:n}^{(2)} - \frac{e^{-x_o}}{F_o}(n-1)\mu_{1:n-1}^{(2)} + (n-2+\alpha)\mu_{1,2:n} +$$

$$\frac{e^{-x_o}}{F_o}(n-1)\mu_{1,2:n-1} + \frac{\alpha e^{-\alpha x_o}}{G_o}\nu_{1,2:n-1} + \frac{(1-\alpha)}{\alpha G_o}\nu_{1:n-1} +$$

$$\left(1-\frac{1}{G_o}\right)\nu_{1:n-1}^{(2)} - (n-1)(1-\alpha)\; I_7, \tag{2.3.5}$$

where $I_7 = \int_o^{x_o} x\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-2}\frac{\bar{e}^{x(\alpha+1)}}{\alpha G_o F_o}\,dx.$

This can also be written as

$$I_7 = \int_o^{x_o} x\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-1}\frac{\bar{e}^{\alpha x}}{\alpha G_o} + \frac{e^{-x_o}}{F_o}\int_o^{x_o} x\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-2}\frac{\bar{e}^{\alpha x}}{\alpha G_o}\,dx.$$

From equation (2.1.5), we get

$$\mu_{1:n}^{(2)} = \int_o^{x_o} 2x\left[1 - F_{1:n-1}(x) - \left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-1}\left(\frac{1-\bar{e}^{\alpha x}}{G_o}\right)\right]dx.$$

On using equation (2.1.7), it gives

$$\mu_{1:n}^{(2)} = \nu_{1:n-1}^{(2)} - \int_o^{x_o} 2x\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-1}\frac{1}{G_o}\,dx + \int_o^{x_o} 2x\left(1-\frac{1-\bar{e}^{x}}{F_o}\right)^{n-1}\frac{\bar{e}^{\alpha x}}{G_o}\,d$$

Again using (2.1.7), we have

$$\mu_{1:n}^{(2)} = \nu_{1:n-1}^{(2)} - \frac{1}{G_o} \nu_{1:n-1}^{(2)} + \int_0^{x_o} 2x\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-1} \frac{e^{-\alpha x}}{G_o}\, dx,$$

which gives

$$\int_0^{x_o} 2x\left(1 - \frac{1-e^{-x}}{F_o}\right)^{n-1} \frac{e^{-\alpha x}}{G_o}\, dx = \mu_{1:n}^{(2)} + \frac{e^{-\alpha x_o}}{G_o} \nu_{1:n-1}^{(2)}.$$

Using this equation, I_7 can be rewritten as

$$I_7 = \frac{1}{2\alpha} \left[\mu_{1:n}^{(2)} + \frac{e^{-\alpha x_o}}{G_o} \nu_{1:n-1}^{(2)} + \frac{e^{-x_o}}{F_o} \left\{\mu_{1:n-1}^{(2)} + \frac{e^{-\alpha x_o}}{G_o} \nu_{1:n-2}^{(2)}\right\}\right].$$

Substituting this value of I_7 in equation (2.3.5), it reduces to

$$\begin{aligned}\mu_{1:n} = & -(n-1)\left(\frac{\alpha+1}{2\alpha}\right)\mu_{1:n}^{(2)} - \frac{e^{-x_o}}{F_o}\left(\frac{\alpha+1}{2\alpha}\right)(n-1)\mu_{1:n-1}^{(2)} + (n-2+\alpha)\mu_{1,2:n} + \\ & \frac{e^{-x_o}}{F_o}(n-1)\mu_{1,2:n-1} + \frac{\alpha e^{-\alpha x_o}}{G_o} \nu_{1,2:n-1} + \frac{(1-\alpha)}{\alpha G_o} \nu_{1:n-1} - \\ & \left[1 + \frac{(n-1)(1-\alpha)}{2\alpha}\right] \frac{e^{-\alpha x_o}}{G_o} \nu_{1:n-1}^{(2)} - \frac{(n-1)(1-\alpha)}{2\alpha} \frac{e^{-x_o}}{F_o} \frac{e^{-\alpha x_o}}{G_o} \nu_{1:n-2}^{(2)}.\end{aligned}$$

On rearranging the terms, we get

$$\begin{aligned}\mu_{1,2:n} = & \frac{1}{(n-2+\alpha)} \left[\mu_{1:n} + \frac{(n-1)(\alpha+1)}{2\alpha} \mu_{1:n}^{(2)} + \frac{(n-1)(\alpha+1)}{2\alpha} \frac{e^{-x_o}}{F_o} \mu_{1:n-1}^{(2)}\right. \\ & - (n-1) \frac{e^{-x_o}}{F_o} \mu_{1,2:n-1} - \frac{\alpha e^{-\alpha x_o}}{G_o} \nu_{1,2:n-1} - \frac{(1-\alpha)}{\alpha G_o} \nu_{1:n-1} \\ & + \left(1 + \frac{(n-1)(1-\alpha)}{2\alpha}\right) \frac{e^{-\alpha x_o}}{G_o} \nu_{1:n-1}^{(2)} + \frac{(n-1)(1-\alpha)}{2\alpha} \frac{e^{-x_o}}{F_o} \\ & \left. \cdot \frac{e^{-\alpha x_o}}{G_o} \nu_{1:n-2}^{(2)}\right],\end{aligned}$$

which is the required result.

The proof of equation (2.3.2), is similar to the proof of equation (2.3.1). We now start with

$$\mu_{n-1:n} = E(X_{n-1:n}\, X^{o}_{n:n}).$$

On using equation (2.1.3) and equation (2.1.8), we have

$$\mu_{n-1:n} = (n-1)(n-2) \iint_W x\left(\frac{1-e^{-x}}{F_o}\right)^{n-3} \left(\frac{1-e^{-\alpha x}}{G_o}\right) \frac{e^{-x}}{F_o} \frac{e^{-y}}{F_o}\, dx\, dy +$$

$$(n-1)\left[\iint_W x\left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{\alpha e^{-\alpha x}}{G_o} \frac{e^{-y}}{F_o}\, dx\, dy + \iint_W x\left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{e^{-x}}{F_o} \frac{\alpha e^{-\alpha y}}{G_o}\, dx\, dy\right].$$

Proceeding in a similar manner as in the proof of equation (2.3.1), we obtain

$$\mu_{n-1:n} = (n-1)(n-2)\left[\frac{x_o e^{-x_o}}{F_o} \int_0^{x_o} x\left(\frac{1-e^{-x}}{F_o}\right)^{n-3} \left(\frac{1-e^{-\alpha x}}{G_o}\right) \frac{e^{-x}}{F_o}\, dx - \int_0^{x_o} x^2\left(\frac{1-e^{-x}}{F_o}\right)^{n-3} \left(\frac{1-e^{-\alpha x}}{G_o}\right) \frac{e^{-2x}}{F_o^2}\, dx + \iint_W xy\left(\frac{1-e^{-x}}{F_o}\right)^{n-3}\right.$$

$$\left. \cdot\left(\frac{1-e^{-\alpha x}}{G_o}\right) \frac{e^{-x}}{F_o} \frac{e^{-y}}{F_o}\, dx\, dy\right] + (n-1)\left[\frac{x_o e^{-x_o}}{F_o} \int_0^{x_o} x\left(\frac{1-e^{-x}}{F_o}\right)^{n-2}\right.$$

$$\cdot \frac{\alpha e^{-\alpha x}}{G_o}\, dx - \int_0^{x_o} x^2\left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{\alpha e^{-\alpha x}}{G_o} \frac{e^{-x}}{F_o}\, dx +$$

$$\left. \iint_W xy\left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{\alpha e^{-\alpha x}}{G_o} \frac{e^{-y}}{F_o}\, dx\right] + (n-1)\left[\frac{\alpha x_o e^{-\alpha x_o}}{G_o} \int_0^{x_o} x\left(\frac{1-e^{-x}}{F_o}\right)^{n-}\right.$$

$$x \frac{e^{-x}}{F_o}\, dx - \alpha \int_0^{x_o} x^2\left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{e^{-x}}{F_o} \frac{e^{-\alpha x}}{G_o}\, dx +$$

$$\left. \alpha^2 \iint_W xy\left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{e^{-x}}{F_o} \frac{e^{-\alpha y}}{G_o}\, dx\, dy\right].$$

Using equations (2.1.4), (2.1.6) and (2.1.8), it simplifies to

$$\mu_{n-1:n} = \frac{(n-1)x_o e^{-x_o}}{F_o}\, \mu_{n-1:n-1} + \frac{\alpha x_o e^{-\alpha x_o}}{G_o}\, \nu_{n-1:n-1} - \frac{(n-1)}{F_o}\, \mu^{(2)}_{n-1:n-1}$$

$$+ (n-1)\mu^{(2)}_{n:n} + \mu_{n-1,n} - \frac{1}{G_o} \nu^{(2)}_{n-1:n-1} - \frac{(1+\alpha x_o)}{\alpha G_o} (\alpha-1)$$

$$\cdot e^{-\alpha x_o} \nu_{n-1:n-1} + (n-1) \frac{(\alpha-1)}{\alpha} I_8 , \tag{2.3.6}$$

where

$$I_8 = \int_0^{x_o} x \left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{e^{-\alpha x}}{G_o} \frac{e^{-x}}{F_o} dx.$$

We can also write I_8 as

$$I_8 = \int_0^{x_o} x\left(\frac{1-e^{-x}}{F_o}\right)^{n-2} \frac{e^{-\alpha x}}{G_o F_o} dx - \int_0^{x_o} x\left(\frac{1-e^{-x}}{F_o}\right)^{n-1} \frac{e^{-\alpha x}}{G_o} dx.$$

Using equations (2.1.5) and (2.1.7) for k = 2, we get

$$I_8 = \frac{1}{2}\left[\frac{1}{F_o} \left\{\mu^{(2)}_{n-1:n-1} - x_o^2 + \frac{1}{G_o} (x_o^2 - \nu^{(2)}_{n-2:n-2})\right\} - \mu^{(2)}_{n:n} + x_o^2 - \frac{1}{G_o} (x_o^2 - \nu^{(2)}_{n-1:n-1})\right].$$

Substituting this value in equation (2.3.6) and simplifying, it gives

$$\mu_{n-1:n} = (n-1) \frac{x_o e^{-x_o}}{F_o} \mu_{n-1:n-1} + \left(\frac{1}{\alpha} + x_o - 1\right) \frac{e^{-\alpha x_o}}{G_o} \nu_{n-1:n-1} +$$

$$\frac{(n-1)}{F_o}\left(\frac{\alpha-1}{2\alpha} - 1\right)\mu^{(2)}_{n-1:n-1} + (n-1)\left(1 - \frac{\alpha-1}{2\alpha}\right)\mu^{(2)}_{n:n} +$$

$$\mu_{n-1,n:n} - \frac{1}{G_o} \left(1 - \frac{(n-1)(\alpha-1)}{2\alpha}\right)\nu^{(2)}_{n-1:n-1} -$$

$$\frac{(n-1)(\alpha-1)}{2\alpha G_o} \nu^{(2)}_{n-2:n-2} + \frac{(n-1)(\alpha-1)}{2\alpha} x_o^2 \frac{e^{-\alpha x_o} e^{-x_o}}{F_o G_o} .$$

Now rearranging the terms, we obtain

$$\mu_{n-1,n:n} = \mu_{n-1:n} - \frac{(n-1)x_o e^{-x_o}}{F_o} \mu_{n-1:n-1} - \left(\frac{1}{\alpha} + x_o - 1\right)\frac{e^{-\alpha x_o}}{G_o} \nu_{n-1:n-1}$$

$$- \frac{(n-1)}{F_o}\left(\frac{\alpha-1}{2\alpha} - 1\right)\mu^{(2)}_{n-1:n-1} - (n-1)\left(1 - \frac{\alpha-1}{2\alpha}\right)\mu^{(2)}_{n:n} +$$

$$\frac{(n-1)(\alpha-1)}{2\alpha G_o}\,\nu^{(2)}_{n-2:n-2} + \frac{1}{G_o}\left(1 - \frac{(n-1)(\alpha-1)}{2\alpha}\right)\nu^{(2)}_{n-1:n-1} -$$

$$\frac{(n-1)(\alpha-1)x_o^2}{2\alpha}\,\frac{e^{-\alpha x_o}}{G_o}\,\frac{e^{-x}}{F_o}\,.$$

Using equation (2.1.10) and simplifying, we finally obtain

$$\mu_{n-1,n:n} = \mu_{n-1:n} - (n-1)\,\frac{x_o e^{-x_o}}{F_o}\,\mu_{n-1:n-1} + \frac{(n-1)(\alpha+1)}{2\alpha}\,\frac{\mu^{(2)}_{n-1:n-1}}{F_o}\,.$$

$$\frac{(n-1)(\alpha+1)}{2\alpha}\,\mu^{(2)}_{n:n} - \left[\frac{\alpha-1}{\alpha} + e^{-\alpha x_o}\left(\frac{1}{\alpha} - 1 + x_o\right)\right]\frac{\nu_{n-1:n-1}}{G_o}$$

$$+ \frac{\nu^{(2)}_{n-1:n-1}}{G_o} - \frac{(n-1)(1-\alpha)}{2\alpha}\,x_o^2\,\frac{e^{-x_o}}{F_o}\,.$$

This completes the proof of the theorem.

For the remainder of this section we consider the untruncated case when $x_o = \infty$. Similar to the case of single order statistic moments, we now have the following result.

<u>COROLLARY 2.3.1</u>: For samples containing a single outlier from an exponential distribution, we have for $n = 3,4,\ldots,$

$$\mu_{1,2:n} = \frac{1}{(n-2+\alpha)}\left[\mu_{1:n} + \frac{(n-1)(\alpha+1)}{2\alpha}\,\mu^{(2)}_{1:n} - \frac{(1-\alpha)}{\alpha}\,\nu_{1:n-1}\right], \tag{2.3.7}$$

and for $n = 2,3,\ldots,$

$$\mu_{n-1,n:n} = \mu_{n-1:n} + \frac{(n-1)(\alpha+1)}{2\alpha}\left(\mu^{(2)}_{n-1:n-1} - \mu^{(2)}_{n:n}\right) + \frac{(1-\alpha)}{\alpha}\,\nu_{n-1:n-1}$$

$$+ \nu^{(2)}_{n-1:n-1}\,. \tag{2.3.8}$$

<u>Proof</u>: The proof follows immediately on taking limit as $x_o \to \infty$ in equations (2.3.1) and (2.3.2).

We next derive a recurrence relation for $\mu_{r,s:n}$ for other values of r and s.

THEOREM 2.3.2: For samples from exponential distribution, containing one outlier, we have

$$\text{(a)}\quad \mu_{r,r+1:n} = (n-r-1+\alpha)^{-1}\Big[\frac{(n-r-1+\alpha)}{(n-r)}\,\mu_{r:n} + \Big(n-r-\frac{1}{2}+\frac{\alpha}{2}\Big)\mu^{(2)}_{r:n} - \frac{(1-\alpha)}{2}\,\nu^{(2)}_{r:n-1} + \frac{(1-\alpha)}{(n-r)}\,\nu_{r:n-1}\Big],\quad r = 1,2,\ldots,n-1,$$

and (2.3.9)

$$\text{(b)}\quad \mu_{r,s:n} = \frac{1}{(n-s+\alpha)}\Big[\mu_{r:n} + (n-s+1)\mu_{r,s-1:n} - \frac{(1-\alpha)}{2}\,\mu^{(2)}_{r:n} - (1-\alpha)\mu_{r:n}\sum_{j=o}^{s-r-1}\frac{1}{j+n-s+1} - \frac{(1-\alpha)}{2}\,\nu^{(2)}_{r:n-1} + \frac{(1-\alpha)}{(n-r)}\,\nu_{r:n-1}\Big], \qquad (2.3.10)$$

where $s-r \geq 2$ and $1 \leq r < s \leq n$.

Proof: Here we again use the technique used by Joshi (1982), by writing

$$\mu_{r:n} = E(X_{r:n}\cdot X^{o}_{r+1:n}).$$

From equations (2.1.3) and (2.1.8), we have

$$\mu_{r:n} = \frac{(n-1)!}{(r-2)!(n-r-1)!}\iint_{W_1} x(1-\bar{e}^{x})^{r-2}(\bar{e}^{y})^{n-r}\,\bar{e}^{x}(1-\bar{e}^{\alpha x})\,dx\,dy$$

$$\frac{(n-1)!}{(r-1)!(n-r-2)!}\iint_{W_1} x(1-\bar{e}^{x})^{r-1}(\bar{e}^{y})^{n-r-1+\alpha}\,\bar{e}^{x}\,dx\,dy$$

$$\frac{(n-1)!}{(r-1)!(n-r-1)!}\iint_{W_1} x(1-\bar{e}^{x})^{r-1}(\bar{e}^{y})^{n-r-1+\alpha}\,\alpha\bar{e}^{x}\,dx\,dy$$

$$+\frac{(n-1)!}{(r-1)!(n-r-1)!}\iint_{W_1} x(1-\bar{e}^{x})^{r-1}\,\alpha\bar{e}^{\alpha x}(\bar{e}^{y})^{n-r}\,dx\,dy,$$

where $w_1 = \{(x,y): 0 < x < y < \infty\}$ is the region of

Here for $r = 1$, first term drops out, and for $r = n-1$, second term drops out.

This can be rewritten as

$$\mu_{r:n} = \frac{(n-1)!}{(r-2)!(n-r-1)!} \int_0^\infty x(1-e^{-x})^{r-2} (1-e^{-\alpha x}) e^{-x} J_{n-r}(x)$$

$$\frac{(n-1)!}{(r-1)!(n-r-2)!} \int_0^\infty x(1-e^{-x})^{r-1} e^{-x} J_{n-r-1+\alpha}(x)\, dx$$

$$\frac{(n-1)!}{(r-1)!(n-r-1)!} \int_0^\infty x(1-e^{-x})^{r-1} \alpha e^{-x} J_{n-r-1+\alpha}(x)\, dx$$

$$\frac{(n-1)!}{(r-1)!(n-r-1)!} \int_0^\infty x(1-e^{-x})^{r-1} \alpha e^{-\alpha x} J_{n-r}(x)\, dx,$$

where

$$J_k(x) = \int_x^\infty e^{-ky}\, dy$$

$$= - x\, e^{-kx} + k \int_x^\infty y\, e^{-ky}\, dy.$$

Substituting this value in equation (2.3.11), it gives

$$\mu_{r:n} = \frac{(n-1)!}{(r-2)!(n-r-1)!} \Big[- \int_0^\infty x^2(1-e^{-x})^{r-2} e^{-(n-r+1)x} (1-e^{-\alpha x})\, dx +$$

$$\int_0^\infty \int_x^\infty xy(1-e^{-x})^{r-2} e^{-x}(1-e^{-\alpha x}) e^{-(n-r)y} (n-r)\, dy\, dx\Big] +$$

$$\frac{(n-1)!}{(r-1)!(n-r-2)!} \Big[- \int_0^\infty x^2(1-e^{-x})^{r-1} e^{-(n-r+\alpha)x}\, dx +$$

$$\int_0^\infty \int_x^\infty xy(1-e^{-x})^{r-1} e^{-x} e^{-(n-r-1+\alpha)y} (n-r-1+\alpha)\, dy\, dx\Big] +$$

$$\frac{(n-1)!}{(r-1)!(n-r-1)!} \Big[- \int_0^\infty x^2(1-e^{-x})^{r-1} \alpha e^{-x} e^{-(n-r-1+\alpha)x}\, dx +$$

$$\int_0^\infty \int_x^\infty xy(1-e^{-x})^{r-1} e^{-x} \alpha e^{-(n-r-1+\alpha)y} (n-r-1+\alpha))\, dy\, dx -$$

$$\int_0^\infty x^2(1-e^{-x})^{r-1}\,\alpha e^{-\alpha x}\,e^{-(n-r)x}\,dx + \int_0^\infty\int_x^\infty xy(1-e^{-x})^{r-1}\,\alpha e^{-\alpha x}$$

$$\cdot e^{-(n-r)y}\,(n-r)\,dy\,dx].$$

Using equations (2.1.4) and (2.1.8), we have

$$\mu_{r:n} = (n-r-1+\alpha)\mu_{r,r+1:n} + \frac{(1-\alpha)\ (n-1)!}{(r-2)!(n-r-1)!}\int_0^\infty\int_x^\infty xy(1-e^{-x})^{r-2}$$

$$\cdot e^{-x}(1-e^{-\alpha x})\,e^{-(n-r)y}\,dy\,dx - (n-r)\mu^{(2)}_{r:n} +$$

$$\frac{(1-\alpha)\ (n-1)!}{(r-1)!(n-r-1)!}\int_0^\infty\int_x^\infty xy(1-e^{-x})^{r-1}\,\alpha e^{-\alpha x}\,e^{-(n-r)y}\,dy\,dx +$$

$$\frac{(1-\alpha)\ (n-1)!}{(r-1)!(n-r-1)!}\left[\int_0^\infty x^2(1-e^{-x})^{r-1}\,e^{-(n-r)x}\,e^{-\alpha x}\,dx\right].$$

For simplifying, we write it as

$$\mu_{r:n} = (n-r-1+\alpha)\ \mu_{r,r+1:n} - (n-r)\mu^{(2)}_{r:n} + (1-\alpha)\left[\frac{(n-1)!}{(r-2)!(n-r-1)!}\right.$$

$$\cdot\int_0^\infty x(1-e^{-x})^{r-2}\,(1-e^{-\alpha x})e^{-x}\,J_{n-r}(x)\,dx + \frac{(n-1)!}{(r-1)!(n-r-1)!}$$

$$\cdot\left\{\int_0^\infty x(1-e^{-x})^{r-1}\,\alpha e^{-\alpha x}\,J_{n-r}(x)\,dx + \int_0^\infty x^2(1-e^{-x})^{r-1}\,e^{-(n-r)x}\right.$$

$$\left.\left.\cdot\ e^{-\alpha x}\,dx\right\}\right],$$

where

$$J_{n-r}(x) = \int_{x}^\infty y\,e^{-(n-r)y}\,dx$$

$$= \frac{x\,e^{-(n-r)x}}{(n-r)} + \frac{e^{-(n-r)x}}{(n-r)^2}\ .$$

Substituting this value in $\mu_{r:n}$ and using equation (2.1.4), we get

$$(n-r-1+\alpha)\mu_{r,r+1:n} = \mu_{r:n} + (n-r)\mu^{(2)}_{r:n} - (1-\alpha)\mu^{(2)}_{r:n} - \frac{(1-\alpha)}{(n-r)}\,\mu_{r:n} +$$

$$\frac{(1-\alpha)\ (n-1)!}{(r-1)!(n-r)!} \int_0^\infty x(1-\bar{e}^x)^{r-1}\ \bar{e}^{(n-r+\alpha)x}\ dx. \qquad (2.3.12)$$

From equation (2.1.4) for k = 2, we have

$$\mu_{r:n}^{(2)} = \int_0^\infty 2x[1 - F_{r:n-1}(x) - \binom{n-1}{r-1}(1-\bar{e}^x)^{r-1}\ \bar{e}^{(n-r)x}(1-\bar{e}^{\alpha x})]dx.$$

Using equations (2.1.6) and (2.1.7), this gives

$$\mu_{r:n}^{(2)} = \nu_{r:n-1}^{(2)} - \frac{2\nu_{r:n-1}}{(n-r)} + 2\binom{n-1}{r-1} \int_0^\infty x\ \bar{e}^{(n-r+\alpha)x}\ (1-\bar{e}^x)^{r-1}\ dx.$$

Substituting the value of the integral in equation (2.3.12), we get

$$\mu_{r,r+1:n} = \frac{1}{(n-r-1+\alpha)}\ [\frac{(n-r-1+\alpha)}{(n-r)}\ \mu_{r:n} + (n-r- \frac{1-\alpha}{2})\mu_{r:n}^{(2)} -$$
$$\frac{(1-\alpha)}{2}\ \nu_{r:n-1}^{(2)} + \frac{(1-\alpha)}{(n-r)}\ \nu_{r:n-1}],$$

which completes the proof of part (a).

The proof of part (b) of the theorem is again similar to the proof of part (a). We now write

$$\mu_{r:n} = E(X_{r:n}\ X_{s:n}^{o}) \quad \text{for} \quad s \geq r+2.$$

Using equations (2.1.3) and (2.1.8), we have

$$\mu_{r:n} = \frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!} \iint_{W_1} x\ (1-\bar{e}^x)^{r-2}\ \bar{e}^x(\bar{e}^x - \bar{e}^y)^{s-r-1}$$
$$\cdot(1-\bar{e}^{\alpha x})\bar{e}^{(n-s+1)y}\ dx\ dy + \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s-1)!} \iint_{W_1} x$$
$$\cdot(1-\bar{e}^x)^{r-1}\ (\bar{e}^x-\bar{e}^y)^{s-r-1}\ \bar{e}^x\ \bar{e}^{(n-s+\alpha)y}\ dx\ dy +$$
$$\frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!}[\alpha \iint_{W_1} x(1-\bar{e}^x)^{r-1}\ (\bar{e}^x-\bar{e}^y)^{s-r-1}\ \bar{e}^x$$
$$\cdot\bar{e}^{(n-s+\alpha)y}\ dx\ dy + \alpha \iint_{W_1} x(1-\bar{e}^x)^{r-1}(\bar{e}^x-\bar{e}^y)^{s-r-1}\ \bar{e}^{\alpha x}$$

$$\cdot \bar{e}^{(n-s+1)y} \, dx \, dy] + \frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!} \cdot$$

$$\cdot \iint_{W_1} x(1-\bar{e}^{x})^{r-1}(\bar{e}^{x}-\bar{e}^{y})^{s-r-2}(\bar{e}^{\alpha x}-\bar{e}^{\alpha y}) \, \bar{e}^{x} \, \bar{e}^{(n-s+1)y} \, dx \, dy.$$

For simplifying, we write $\mu_{r:n}$ as

$$\mu_{r:n} = I_1 + I_2 + I_3 + I_4 + I_5, \tag{2.3.13}$$

where for $r = 1$, I_1 is zero and for $s = n$, I_2 is zero. First consider I_1

$$I_1 = \frac{(n-1)!}{(r-2)!(n-s)!(s-r-1)!} \int_0^{\infty} x(1-\bar{e}^{x})^{r-2} \, (1-\bar{e}^{\alpha x})\bar{e}^{x} \, J_1(x) \, dx,$$

where

$$J_1(x) = \int_x^{\infty} \bar{e}^{(n-s+1)y} \, (\bar{e}^{x} - \bar{e}^{y})^{s-r-1} \, dy.$$

Integrating $J_1(x)$ by parts, treating 1 for integration and $\bar{e}^{(n-s+1)y} \, (\bar{e}^{x}-\bar{e}^{y})^{s-r-1}$ for differentiation, we get

$$J_1 = \int_x^{\infty} y[(n-s+1)\bar{e}^{(n-s+1)y} \, (\bar{e}^{x}-\bar{e}^{y})^{s-r-1} -$$
$$(s-r-1)(\bar{e}^{x} - \bar{e}^{y})^{s-r-2} \, \bar{e}^{(n-s+2)y}] \, dy.$$

Then substituting this value in I_1, it gives

$$I_1 = \frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!} [(n-s+1) \iint_{W_1} xy(1-\bar{e}^{x})^{r-2} \, \bar{e}^{x}(1-\bar{e}^{\alpha x})$$
$$\cdot(\bar{e}^{x} - \bar{e}^{y})^{s-r-1} \, \bar{e}^{(n-s+1)y} \, dx \, dy - (s-r-1) \iint_{W_1} xy(1-\bar{e}^{x})^{r-2}$$
$$\cdot\bar{e}^{x}(1-\bar{e}^{\alpha x})(\bar{e}^{x} - \bar{e}^{y})^{s-r-2} \, \bar{e}^{(n-s+2)y} \, dx \, dy].$$

Writing for $I_2,\ldots,I_5$ also in a similar manner, substituting these values of $I_1,\ldots,I_5$ in equation (2.3.13), and using equations (2.1.3) and (2.1.8), we have

$$\mu_{r:n} = (n-s+\alpha)\mu_{r,s:n} - \frac{(\alpha-1)\ (n-1)!}{(r-2)!(s-r-1)!(n-s)!} \iint_{w_1} xy(1-\bar{e}^{x})^{r-2}$$

$$\cdot\ \bar{e}^{x}(1-\bar{e}^{\alpha x})\bar{e}^{(n-s+1)y}\ (\bar{e}^{x}-\bar{e}^{y})^{s-r-1}\ dx\ dy\ -$$

$$\frac{(\alpha-1)\ (n-1)!}{(r-1)!(s-r-1)!(n-s)!} \iint_{w_1} xy(1-\bar{e}^{x})^{r-1}\ \alpha\bar{e}^{\alpha x}(\bar{e}^{x}-\bar{e}^{y})^{s-r-1}$$

$$\cdot\ \bar{e}^{(n-s+1)y}\ dx\ dy - \frac{(\alpha-1)\ (n-1)!}{(r-1)!(s-r-2)!(n-s)!} \iint_{w_1} xy(1-\bar{e}^{x})^{r-1}$$

$$\cdot\ \bar{e}^{x}(\bar{e}^{x}-\bar{e}^{y})^{s-r-2}\ (\bar{e}^{\alpha x}-\bar{e}^{\alpha y})\ \bar{e}^{(n-s+1)y}\ dx\ dy - (n-s+1)\mu_{r,s-1:}$$

$$+\ \frac{(1-\alpha)\ (n-1)!}{(r-1)!(s-r-2)!(n-s)!} \iint_{w_1} xy(1-\bar{e}^{x})^{r-1}\ (\bar{e}^{x}-\bar{e}^{y})^{s-r-2}$$

$$\cdot\ \bar{e}^{x}\ \bar{e}^{(n-s+1+\alpha)y}\ dx\ dy.$$

On simplifying, it reduces to

$$\mu_{r:n} = (n-s+\alpha)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n} - (\alpha-1)$$

$$\cdot\left[\frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!} \iint_{w_1} xy(1-\bar{e}^{x})^{r-2}\ (\bar{e}^{x}-\bar{e}^{y})^{s-r-1}\right.$$

$$\cdot\ \bar{e}^{x}\ \bar{e}^{(n-s+1)y}\ (1-\bar{e}^{\alpha x})\ dx\ dy + \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!}$$

$$\cdot \iint_{w_1} xy(1-\bar{e}^{x})^{r-1}\ (\bar{e}^{x}-\bar{e}^{y})^{s-r-1}\ \alpha\bar{e}^{\alpha x}\ \bar{e}^{(n-s+1)y}\ dx\ dy\ +$$

$$\frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!} \iint_{w_1} xy(1-\bar{e}^{x})^{r-1}\ \bar{e}^{x}(\bar{e}^{x}-\bar{e}^{y})^{s-r-2}$$

$$\left.\cdot\ \bar{e}^{\alpha x}\ \bar{e}^{(n-s+1)y}\ dx\ dy\right].$$

This can be written as

$$\mu_{r:n} = (n-s+\alpha)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n} - (\alpha-1)[I_6 + I_7 + I_8]. \tag{2.3.14}$$

Now we simplify each I_i separately. I_6, I_7 and I_8 can be written as

$$I_6 = \frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!} \int_0^\infty x(1-\bar{e}^{x})^{r-2}\, \bar{e}^{x}(1-\bar{e}^{\alpha x})\, J_{s-r-1}(x)\, dx,$$

$$I_7 = \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!} \int_0^\infty x(1-\bar{e}^{x})^{r-1}\, \alpha\bar{e}^{\alpha x}\, J_{s-r-1}(x)\, dx,$$

$$I_8 = \frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!} \int_0^\infty x(1-\bar{e}^{x})^{r-1}\, \bar{e}^{x}\, \bar{e}^{\alpha x}\, J_{s-r-2}(x)\, dx,$$

where $\quad J_k(x) = \int_x^\infty y(\bar{e}^{x} - \bar{e}^{y})^k\, \bar{e}^{(n-s+1)y}\, dy.$

Transform y to t by a transformation $\bar{e}^{y} = \bar{e}^{x}t$. This gives

$$J_k(x) = \int_0^1 (x-\log t)(1-t)^k\, t^{n-s}\, \bar{e}^{(n-s+1+k)x}\, dt$$

$$= \bar{e}^{(n-s)x}[xB(n-s+1,k+1) + B(n-s+1,k+1) \sum_{j=0}^{k} \frac{1}{j+n-s+1}]\bar{e}^{(k+1)x}$$

on using a result given by Gross et al. (1986).

Substituting for $J_{s-r-1}(x)$ and $J_{s-r-2}(x)$ in I_6, I_7 and I_8, we get

$$I_6 = \frac{(n-1)!}{(r-2)!(n-r)!} [\int_0^\infty x^2(1-\bar{e}^{x})^{r-2}\, \bar{e}^{(n-r+1)x}\, (1-\bar{e}^{\alpha x})\, dx +$$
$$\int_0^\infty x(1-\bar{e}^{x})^{r-2}\, \bar{e}^{(n-r+1)x}\, (1-\bar{e}^{\alpha x}) \sum_{j=0}^{s-r-1} \frac{1}{j+n-s+1}\, dx],$$

$$I_7 = \frac{\alpha(n-1)!}{(r-1)!(n-r)!} [\int_0^\infty x^2(1-\bar{e}^{x})^{r-1}\, \bar{e}^{(n-r+\alpha)x}\, dx +$$
$$\int_0^\infty x(1-\bar{e}^{x})^{r-1}\, \bar{e}^{(n-r+\alpha)x}\, dx \sum_{j=0}^{s-r-1} \frac{1}{j+n-s+1}],$$

$$I_8 = \frac{(n-1)!}{(r-1)!(n-r-1)!} [\int_0^\infty x^2(1-\bar{e}^{x})^{r-1}\, \bar{e}^{(n-r+\alpha)x}\, dx +$$
$$\int_0^\infty x(1-\bar{e}^{x})^{r-1}\, \bar{e}^{(n-r+\alpha)x}\, dx \sum_{j=0}^{s-r-2} \frac{1}{j+n-s+1}].$$

Substituting these values in equation (2.3.14) and using equation (2.1.5) for $k = 1,2$, we obtain

$$\mu_{r:n} = (n-s+\alpha)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n} - (\alpha-1)[\mu^{(2)}_{r:n} + \mu_{r:n}\sum_{j=o}^{s-r-1}\frac{1}{j+n-s+1} - \binom{n-1}{r-1}\int_o^\infty x(1-\bar{e}^{x})^{r-1}\,\bar{e}^{(n-r+\alpha)x}\,dx].$$

Now using equation (2.1.5) again, and equation (2.1.7) for k = 2 and equation (2.1.6), it reduces to

$$\mu_{r:n} = (n-s+\alpha)\mu_{r,s:n} - (n-s+1)\mu_{r,s-1:n} - (\alpha-1)[\mu^{(2)}_{r:n} + \mu_{r:n}\sum_{j=o}^{s-r-1}\frac{1}{j+n-s+1} - \frac{1}{2}\{\mu^{(2)}_{r:n} - \nu^{(2)}_{r:n-1} + \frac{2}{(n-r)}\,\nu_{r:n-1}\}],$$

which gives

$$\mu_{r,s:n} = \frac{1}{n-s+\alpha}\,[\mu_{r:n} + (n-s+1)\mu_{r,s-1:n} + \frac{(\alpha-1)}{2}\,\mu^{(2)}_{r:n} + (\alpha-1)\sum_{j=o}^{s-r-1}\frac{1}{j+n-s+1}\,\mu_{r:n} + \nu^{(2)}_{r:n-1}\,\frac{(\alpha-1)}{2} - \frac{(\alpha-1)}{(n-r)}\,\nu_{r:n-1}]$$

After simplification it gives (2.3.10). This completes the proo of the theorem.

Here if we substitute r = 1, s = 2, we get equation (2.3.7) and for r = n-1, s = n, it reduces to equation (2.3.8).

<u>COROLLARY 2.3.2</u>: For a random sample from exponential distribution, we have

$$\nu_{r,s:n} - \nu_{r,s-1:n} = \frac{1}{n-s+1}\,\nu_{r:n}.$$

<u>Proof</u>: Substituting $\alpha = 1$ in equation (2.3.10), the corollary follows.

<u>2.4 Evaluation of moments of order statistics</u>

In this section, we give a method of obtaining moments o order statistics from truncated exponential distribution in

one outlier situation. First we evaluate single moments by recurrence relations given in equations (2.2.1) and (2.2.2). For this, we need $\nu_{r:n}^{(k)}$, $k = 1,2,\ldots,$ as input. These moments are calculated in Saleh <u>et al</u>. (1975). However, we calculate these moments by using recursive equation (2.1.10) obtained by Joshi (1978). Then for $k = 1$, relation (2.2.2) gives $\mu_{1:1}^{(1)}$, $\mu_{2:2}^{(1)},\ldots,\mu_{n:n}^{(1)}$ and equation (2.2.1) gives $\mu_{1:2}^{(1)}$, $\mu_{1:3}^{(1)}$, $\mu_{2:3}^{(1)}\ldots$, $\mu_{1:n}^{(1)}$, $\mu_{2:n}^{(1)}$, $\mu_{3:n}^{(1)},\ldots,\mu_{n-1:n}^{(1)}$. For $k = 2$, equation (2.2.2) gives $\mu_{1:1}^{(2)}$, $\mu_{2:2}^{(2)},\ldots,\mu_{n:n}^{(2)}$ and equation (2.2.1) gives $\mu_{1:2}^{(2)}$, $\mu_{1:3}^{(2)}$, $\mu_{2:3}^{(2)},\ldots$ $\mu_{1:n}^{(2)}$, $\mu_{2:n}^{(2)},\ldots,\mu_{n-1:n}^{(2)}$ and so on. Thus all the moments can be obtained with sufficient accuracy by means of a simple computer program.

Now if we have $\nu_{r:n}^{(k)}$, $\mu_{r:n}^{(k)}$ for $k = 1,2$ and $r = 1,\ldots,n$, we can evaluate $\mu_{n-1,n:n}$ by using recurrence relation (2.3.2). Application of equation (2.3.1) for $\mu_{1,2:n}$ is somewhat complicated because it needs more initial values.

Consequently for calculating all product moments we use numerical integration technique as follows:

Using equations (2.1.3) and (2.1.8) for $1 \leq r < s \leq n$, we have

$$\mu_{r,s:n} = \frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!} \int_0^{x_0}\int_0^{y} xy\left(\frac{1-e^{-x}}{F_0}\right)^{r-2} \frac{e^{-x}}{F_0}\left(\frac{e^{-x}-e^{-y}}{F_0}\right)^{s-r-1}$$

$$\cdot\left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-s} \frac{e^{-y}}{F_0}\left(\frac{1-e^{-\alpha x}}{G_0}\right) dx\, dy + \frac{(n-1)!}{(r-1)!(s-r-1)!(n-1-s)!}$$

$$\cdot\int_0^{x_0}\int_0^{y} xy\left(\frac{1-e^{-x}}{F_0}\right)^{r-1} \left(\frac{e^{-x}-e^{-y}}{F_0}\right)^{s-r-1} \frac{e^{-x}}{F_0}\left(1 - \frac{1-e^{-y}}{F_0}\right)^{n-s-1} \frac{e^{-y}}{F_0}$$

$$\cdot\left(1-\frac{1-\bar{e}^{\alpha y}}{G_o}\right) dx\, dy + \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!}\int_o^{x_o}\int_o^{y} xy\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r}$$

$$\cdot\left(\frac{\bar{e}^{x}-\bar{e}^{y}}{F_o}\right)^{s-r-1}\left(1-\frac{1-\bar{e}^{y}}{F_o}\right)^{n-s}\frac{\bar{e}^{x}}{F_o}\,\frac{\alpha\bar{e}^{\alpha y}}{G_o}\, dx\, dy +$$

$$\frac{(n-1)!}{(r-1)!(s-r-1)!(n-s)!}\int_o^{x_o}\int_o^{y} xy\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1}\left(\frac{\bar{e}^{x}-\bar{e}^{y}}{F_o}\right)^{s-r-1}$$

$$\cdot\left(1-\frac{1-\bar{e}^{y}}{F_o}\right)^{n-s}\frac{\alpha\bar{e}^{\alpha x}}{G_o}\,\frac{\bar{e}^{y}}{F_o}\, dx\, dy + \frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!}$$

$$\cdot\int_o^{x_o}\int_o^{y} xy\left(\frac{1-\bar{e}^{x}}{F_o}\right)^{r-1}\left(\frac{\bar{e}^{x}-\bar{e}^{y}}{F_o}\right)^{s-r-2}\left(\frac{\bar{e}^{\alpha x}-\bar{e}^{\alpha y}}{G_o}\right)\left(1-\frac{1-\bar{e}^{y}}{F_o}\right)^{n-s}$$

$$\cdot\frac{\bar{e}^{x}}{F_o}\,\frac{\bar{e}^{y}}{F_o}\, dx\, dy.$$

After some simplification, we get

$$\mu_{r,s:n} = \frac{1}{F_o^{n-1}G_o}\Big[\frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!}\int_o^{x_o}\int_o^{y} xy(1-\bar{e}^{x})^{r-2}$$

$$\cdot\ (\bar{e}^{x}-\bar{e}^{y})^{s-r-1}(F_o-1+\bar{e}^{y})^{n-s}\,\bar{e}^{x}\bar{e}^{y}\, dx\, dy -$$

$$\frac{(n-1)!}{(r-2)!(s-r-1)!(n-s)!}\int_o^{x_o}\int_o^{y} xy(1-\bar{e}^{x})^{r-2}(\bar{e}^{x}-\bar{e}^{y})^{s-r-1}$$

$$\cdot\ (F_o-1+\bar{e}^{y})^{n-s}\,\bar{e}^{x(\alpha+1)}\,\bar{e}^{y}\, dx\, dy + \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s-1)!}$$

$$\cdot\int_o^{x_o}\int_o^{y} xy(1-\bar{e}^{x})^{r-1}(\bar{e}^{x}-\bar{e}^{y})^{s-r-1}(F_o-1+\bar{e}^{y})^{n-s-1}\,\bar{e}^{x}\,\bar{e}^{y}$$

$$\cdot\ (G_o-1)\, dx\, dy - \frac{(n-1)!}{(r-1)!(s-r-1)!(n-s-1)!}\int_o^{x_o}\int_o^{y} xy(1-\bar{e}^{x})^{r-1}$$

$$\cdot\ (\bar{e}^{x}-\bar{e}^{y})^{s-r-1}(F_o-1+\bar{e}^{y})^{n-s-1}\,\bar{e}^{x}\bar{e}^{y(\alpha+1)}\, dx\, dy +$$

$$\frac{(n-1)!\alpha}{(r-1)!(s-r-1)!(n-s)!}\int_o^{x_o}\int_o^{y} xy(1-\bar{e}^{x})^{r-1}(\bar{e}^{x}-\bar{e}^{y})^{s-r-1}$$

$$\cdot\ (F_o-1+\bar{e}^{y})^{n-s}\,\bar{e}^{x}\bar{e}^{\alpha y}\, dx\, dy + \frac{(n-1)!\alpha}{(r-1)!(s-r-1)!(n-s)!}$$

$$\cdot \int_0^{x_0} \int_0^{y} xy(1-e^{-x})^{r-1} (e^{-x}-e^{-y})^{s-r-1} (F_0-1+e^{-y})^{n-s} e^{-\alpha x} e^{-y}\, dx\, dy +$$

$$\frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!} \int_0^{x_0} \int_0^{y} xy(1-e^{-x})^{r-1} (e^{-x}-e^{-y})^{s-r-2}$$

$$\cdot (F_0-1+e^{-y})^{n-s} e^{-(\alpha+1)x} e^{-y}\, dx\, dy - \frac{(n-1)!}{(r-1)!(s-r-2)!(n-s)!}$$

$$\cdot \int_0^{x_0} \int_0^{y} xy(1-e^{-x})^{r-1} (e^{-x}-e^{-y})^{s-r-2} (F_0-1+e^{-y})^{n-s} e^{-x}$$

$$\cdot e^{-y(\alpha+1)}\, dx\, dy]. \qquad (2.4.1)$$

These integrals are calculated separately and added to calculate $\mu_{r,s:n}$. All integrals are of the type

$$I(a,b,c,d,f) = \int_0^{x_0} \int_0^{y} xy(1-e^{-x})^{a} (e^{-x}-e^{-y})^{b} [F_0-1+e^{-y}]^{c} e^{-xd} e^{-yf}\, dx\, dy$$

$$= \int \int f(u,v)\, du\, dv \quad \text{(say)}$$

and can be evaluated numerically in a systematic manner by using Gaussian quadrature. For computational purposes, we evaluate these integrals by applying Gauss quadrature formula first to the inner integral and then to the outer integral. This is briefly described in Appendix A. Using this approximation to each integral of equation (2.4.1), we can obtain $\mu_{r,s:n}$.

For the untruncated case, single and product moments are calculated by the method described by Joshi (1972).

Using the Gaussian quadrature, we may also calculate $\mu_{r:n}^{(k)}$, $r = 1(1)n$. Tables 2.4.1 and 2.4.2 give the means and variances-covariances respectively of order statistics obtained by numerical integration using 10 point formula for $n = 5$, $\alpha = .1, .2, .5, 1$ and $x_0 = 1, 2, 3, 4, 5$. These are checked for their accuracy by the following identities:

$$\sum_{r=1}^{n} \mu_{r:n}^{(k)} = (n-1)\, E(X^k) + E(Y^k)$$

and

$$\sum_{r=1}^{n} \sum_{s=1}^{n} \sigma_{r,s:n} = (n-1)\, V(X) + V(Y),$$

where X has the pdf $f(x)$ and Y has the pdf $g(x)$ given in Section 2.1.

The values corresponding to $x_o = \infty$ are also included and are evaluated by the method given by Joshi (1972). As it can be seen from the table that as x_o increases the values of means and variances-covariances also increase and for $x_o = 1,2,$ these values are more or less same for all values of α which means that for small values of x_o, the outlying observation has very little effect on these moments. For $x_o = 10$, sufficient accuracy could not be achieved by the 10 point formula. So we have used the 20 point formula for calculations. These values are also given in Tables 2.4.1 and 2.4.2. For x_o less than 10, the values obtained by 10 point formula and 20 point formula agree upto 4 decimal places atleast. Therefore for larger value of x_o, Gaussian quadrature with higher point formula could be used.

TABLE 2.4.1: Expected values of $X_{r:n}$, $r = 1(1)n$, for $n = 5$ when the sample contains one outlier

α	r \ x_o	1	2	3	4	5	10	∞
.1	1	.1238	.1852	.2136	.2265	.2326	.2403	.2439
	2	.2609	.4053	.4780	.5132	.5307	.5538	.5647
	3	.4138	.6734	.8209	.9002	.9427	1.0035	1.0329
	4	.5856	1.0093	1.2937	1.4759	1.5911	1.7940	1.9008
	5	.7797	1.4414	1.9902	2.4527	2.8598	4.5867	10.2577
.2	1	.1230	.1834	.2111	.2235	.2293	.2362	.2381
	2	.2594	.4012	.4717	.5054	.5217	.5320	.5476
	3	.4118	.6665	.8087	.8835	.9226	.9743	.9892
	4	.5833	.9995	1.2722	1.4414	1.5443	1.7087	1.7597
	5	.7779	1.4307	1.9584	2.3839	2.7365	3.9717	5.4654
.5	1	.1208	.1778	.2028	.2135	.2181	.2220	.2222
	2	.2550	.3886	.4517	.4799	.4924	.5034	.5040
	3	.4057	.6458	.7715	.8323	.8607	.8867	.8881
	4	.5766	.9709	1.2106	1.3434	1.4131	1.4854	1.4897
	5	.7725	1.4008	1.8730	2.2063	2.4329	2.8328	2.8960
1.	1	.1168	.1675	.1878	.1954	.1983	.2000	.2000
	2	.2476	.3671	.4179	.4379	.4455	.4500	.4500
	3	.3957	.6126	.7142	.7567	.7734	.7833	.7833
	4	.5658	.9282	1.1252	1.2183	1.2579	1.2832	1.2833
	5	.7642	1.3594	1.7690	2.0185	2.1553	2.2814	2.2833

TABLE 2.4.2: Variances and covariances of order statistics from truncated exponential distribution when the sample contains one outlier for n = 5

α	r	s \ x_o	1	2	3	4	5	10	∞
.1	1	1	.0127	.0312	.0436	.0502	.0535	.0577	.059
		2	.0111	.0290	.0420	.0493	.0530	.0576	.059
		3	.0091	.0257	.0392	.0474	.0519	.0574	.059
		4	.0067	.0205	.0335	.0426	.0484	.0567	.059
		5	.0037	.0124	.0215	.0282	.0325	.0393	.059
	2	2	.0245	.0691	.1056	.1279	.1401	.1559	.162
		3	.0201	.0613	.0985	.1232	.1373	.1557	.162
		4	.0148	.0490	.0844	.1111	.1284	.1544	.163
		5	.0082	.0296	.0543	.0743	.0877	.1121	.176
	3	3	.0335	.1122	.1954	.2585	.2990	.3592	.384
		4	.0246	.0899	.1677	.2339	.2808	.3586	.388
		5	.0136	.0544	.1085	.1583	.1956	.2747	.459
	4	4	.0367	.1505	.3158	.4899	.6420	.9978	1.175
		5	.0203	.0913	.2055	.3357	.4571	.8121	1.550
	5	5	.0285	.1472	.3910	.7719	1.2946	6.1865	95.939
.2	1	1	.0126	.0307	.0427	.0490	.0521	.0557	.056
		2	.0110	.0286	.0412	.0481	.0516	.0557	.056
		3	.0091	.0254	.0385	.0464	.0506	.0556	.056
		4	.0067	.0205	.0333	.0422	.0476	.0551	.056
		5	.0037	.0126	.0221	.0296	.0347	.0449	.056
	2	2	.0243	.0680	.1032	.1242	.1355	.1493	.152
		3	.0201	.0605	.0966	.1200	.1331	.1493	.153
		4	.0148	.0488	.0836	.1094	.1257	.1487	.154
		5	.0082	.0300	.0556	.0772	.0927	.1255	.162
	3	3	.0334	.1109	.1909	.2501	.2869	.3382	.351
		4	.0246	.0894	.1654	.2285	.2719	.3392	.355
		5	.0137	.0549	.1104	.1626	.2032	.2981	.398
	4	4	.0367	.1499	.3105	.4737	.6106	.8989	.987
		5	.0204	.1220	.2079	.3396	.4626	.8275	1.200
	5	5	.0287	.1493	.3958	.7714	1.2676	5.2392	22.149

contd...

TABLE 2.4.2 (continued)

α	r	s \ x_o	1	2	3	4	5	10	∞
.5	1	1	.0122	.0290	.0396	.0448	.0472	.0493	.04
		2	.0107	.0272	.0384	.0442	.0469	.0493	.04
		3	.0089	.0244	.0363	.0430	.0463	.0492	.04
		4	.0067	.0201	.0323	.0402	.0447	.0492	.04
		5	.0038	.0129	.0232	.0317	.0378	.0479	.04
	2	2	.0238	.0647	.0955	.1127	.1211	.1286	.12
		3	.0198	.0581	.0904	.1097	.1196	.1288	.12
		4	.0147	.0478	.0803	.1027	.1156	.1289	.12
		5	.0083	.0306	.0576	.0810	.0981	.1270	.13
	3	3	.0330	.1067	.1773	.2248	.2511	.2772	.27
		4	.0246	.0877	.1574	.2103	.2428	.2787	.28
		5	.0139	.0560	.1127	.1659	.2065	.2779	.28
	4	4	.0369	.1478	.2942	.4263	.5209	.6495	.65
		5	.0208	.0942	.2101	.3353	.4427	.6584	.69
	5	5	.0295	.1550	.4040	.7505	1.1391	2.6036	3.18
1.	1	1	.0116	.0261	.0342	.0378	.0392	.0400	.04
		2	.0103	.0247	.0334	.0374	.0391	.0400	.04
		3	.0086	.0225	.0321	.0368	.0387	.0400	.04
		4	.0065	.0190	.0293	.0353	.0381	.0400	.04
		5	.0037	.0127	.0225	.0301	.0349	.0399	.04
	2	2	.0229	.0591	.0832	.0948	.0996	.1025	.10
		3	.0192	.0539	.0797	.0931	.0988	.1025	.10
		4	.0145	.0454	.0728	.0892	.0971	.1025	.10
		5	.0083	.0303	.0556	.0758	.0888	.1023	.10
	3	3	.0324	.0999	.1569	.1893	.2039	.2135	.21
		4	.0245	.0840	.1429	.1810	.2001	.2135	.21
		5	.0140	.0559	.1088	.1532	.1823	.2131	.21
	4	4	.0371	.1443	.2707	.3662	.4204	.4632	.46
		5	.0213	.0956	.2047	.3078	.3807	.4621	.46
	5	5	.0307	.1624	.4099	.7121	.9862	1.4465	1.46

CHAPTER 3

SOME IDENTITIES FOR THE MOMENTS OF ORDER STATISTICS IN THE HOMOGENEOUS CASE

3.1 Introduction

Many authors have derived several recurrence relations and identities among the moments of order statistics, for example, see Govindarajulu (1963), Joshi (1971, 1973), Arnold (1977), David (1981), Joshi and Balakrishnan (1981, 1982) for some such relations. An up-to-date summary is given in Balakrishnan et al. (1988). Identities can be used for checking the calculations of moments of order statistics. In proving these relations, mostly it is assumed that the order statistics are from a continuous distribution. But many relations hold for discrete case also (Balakrishnan, 1986).

In this chapter, we assume that we have a random sample $X_1,\ldots,X_n$ of size n from a continuous distribution with pdf f(x) and cdf F(x). The order statistics in this homogeneous case are now denoted by $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$.

The density function of $X_{r:n}$ is given by (David, 1981, p. 9)

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)} (F(x))^{r-1} (1-F(x))^{n-r} f(x) \qquad (3.1.1)$$

and cdf of $X_{r:n}$ is given by

$$F_{r:n}(x) = \sum_{i=r}^{n} \binom{n}{i} F^{i}(x) (1-F(x))^{n-i} \qquad (3.1.2)$$

$$= \frac{1}{B(r,n-r+1)} \int_0^{F(x)} u^{r-1} (1-u)^{n-r} du \qquad (3.1.3)$$

For uniform distribution over $[0,1]$, equation (3.1.2) simply reduces to

$$F_{r:n}(x) = \sum_{i=r}^{n} \binom{n}{i} x^{i}(1-x)^{n-i} \qquad 0 \leq x \leq 1 . \qquad (3.1.4)$$

In Section 3.2, we derive some additional identities for cdf's and moments of order statistics for three cases. These are the distributions for which

(a) moments like $\mu_{i:n}$ for $i = 1,2,\ldots,r$ $(r < n)$ do not exist.

(b) moments like $\mu_{i:n}$ for $i = n-r+1,\ldots,n$ $(r < n)$ do not exist.

(c) moments like $\mu_{i:n}$ for $i = 1,2,\ldots,r,n-r+1,\ldots,n$ $(r < [\frac{n}{2}])$ do not exist.

These cases include the well known Cauchy distribution for which $E(X^k_{r:n}) < \infty$ for $k+1 \leq r \leq n-k$ (Barnett, 1966) and Pareto type distribution. The derived identities may also be applied for other distributions as well.

3.2 Identities when moments of extreme order statistics do not exist

We first derive identities for cdf's of order statistics. Similar results hold for pdf's and moments of order statistics as well.

3.2.1 Case when extremes at one end do not have finite moments

In this section, we generalize the results given by Joshi (1973) and Balakrishnan and Malik (1985). We also apply these results to the following distributions for obtaining some combinatorial identities.

CENTRAL LIBRARY
KANPUR

(i) $$F_1(x) = \begin{cases} -\frac{1}{x} & -\infty < x \le -1 \\ 1 & x > -1 . \end{cases}$$

For this distribution, it is easy to show that

$$\nu_{r:n}^{(k)} = \frac{(-1)^k \, n(n-1) \ldots (n-k+1)}{(r-1) \ldots (r-k)} , \tag{3.2.1}$$

$$k = 1,2,\ldots,n-1 \text{ and } k+1 \le r \le n.$$

(ii) $$F_2(x) = \begin{cases} 0 & x \le 1 \\ 1 - \frac{1}{x} & 1 < x < \infty. \end{cases}$$

For this distribution, we have

$$\nu_{r:n}^{(k)} = \frac{n(n-1) \ldots (n-k+1)}{(n-r) \ldots (n-r-k+1)} , \quad k = 1,2,\ldots,n-1, \tag{3.2.2}$$

and this is finite for $1 \le r \le n-k$.

<u>THEOREM 3.2.1</u>: For an arbitrary continuous cdf $F(x)$,

(i) $$\frac{1}{n+1} \sum_{i=2}^{n+1} F_{i:n+1}(x) \sum_{j=2}^{i} \frac{1}{n-j+2} = \sum_{i=2}^{n+1} \frac{F_{i:i}(x)}{i(i-1)} , \tag{3.2.3}$$

(ii) $$\frac{1}{n+1} \sum_{i=1}^{n} F_{i:n+1}(x) \sum_{j=i}^{n} \frac{1}{j} = \sum_{i=2}^{n+1} \frac{F_{1:i}(x)}{i(i-1)} , \tag{3.2.4}$$

(iii) $$\frac{1}{(n+1) \ldots (n+r)} \sum_{i=r+1}^{n+r} F_{i:n+r}(x) \sum_{j=r+1}^{i} \frac{1}{n+r+1-j} \binom{r-j+i-1}{r-1}$$

$$= \sum_{i=r+1}^{n+r} \frac{F_{i:i}(x)}{i(i-1) \ldots (i-r)} , \quad r \ge 1 , \tag{3.2.5}$$

(iv) $$\frac{1}{(n+1) \ldots (n+r)} \sum_{i=1}^{n} F_{i:n+r}(x) \sum_{j=i}^{n} \frac{1}{j} \binom{j+r-i-1}{r-1}$$

$$= \sum_{i=r+1}^{n+r} \frac{F_{1:i}(x)}{i(i-1) \ldots (i-r)} , \quad r \ge 1 . \tag{3.2.6}$$

Proof: (i) Joshi (1973) has shown that

$$\sum_{i=1}^{n} \frac{1}{n-i+1} F_{i:n}(x) = \sum_{i=1}^{n} \frac{F_{i:i}(x)}{i} . \tag{3.2.7}$$

Substituting for $F_{i:i}(x)$ and $F_{i:n}(x)$ from equation (3.1.4), this yields

$$\sum_{i=1}^{n} \frac{1}{n-i+1} \sum_{r=i}^{n} \binom{n}{r} x^r (1-x)^{n-r} = \sum_{i=1}^{n} \frac{x^i}{i} .$$

Integrating it from 0 to $F(x)$ and using equation (3.1.3), we get

$$\sum_{i=1}^{n} \frac{1}{n+1} F_{i+1:n+1}(x) \sum_{j=1}^{i} \frac{1}{n-j+1} = \sum_{i=1}^{n} \frac{F_{i+1:i+1}(x)}{i(i+1)} .$$

This can be rewritten as

$$\frac{1}{n+1} \sum_{i=2}^{n+1} F_{i:n+1}(x) \sum_{j=2}^{i} \frac{1}{n-j+2} = \sum_{i=2}^{n+1} \frac{F_{i:i}(x)}{i(i-1)} ,$$

which proves equation (3.2.3).

(ii) We start with the second identity given by Joshi (1973), that is

$$\sum_{i=1}^{n} \frac{F_{i:n}(x)}{i} = \sum_{i=1}^{n} \frac{F_{1:i}(x)}{i} . \tag{3.2.8}$$

Substituting for $F_{i:n}(x)$ and $F_{1:i}(x)$ from equation (3.1.4), we get

$$\sum_{i=1}^{n} \frac{1}{i} \sum_{j=i}^{n} \binom{n}{j} x^j (1-x)^{n-j} = \sum_{i=1}^{n} \frac{[1-(1-x)^i]}{i} .$$

Writing $\sum_{j=i}^{n} \binom{n}{j} x^j(1-x)^{n-j}$ as $1 - \sum_{j=o}^{i-1} \binom{n}{j} x^j(1-x)^{n-j}$, and on simplification, this can be written as

$$\sum_{i=1}^{n} \frac{1}{i} \sum_{j=o}^{i-1} \binom{n}{j} x^j (1-x)^{n-j} = \sum_{i=1}^{n} \frac{(1-x)^i}{i} .$$

Proceeding on the same lines as in the proof of equation (3.2.3), we get

$$\frac{1}{n+1} \sum_{i=1}^{n} F_{i:n+1}(x) \sum_{j=i}^{n} \frac{1}{j} = \sum_{i=2}^{n+1} \frac{F_{1:i}(x)}{i(i-1)} ,$$

which is equation (3.2.4).

(iii) To prove equation (3.2.5), we start with equation (3.2.3) which can be rewritten as

$$\sum_{i=2}^{n+1} \frac{F_{i:i}(x)}{i(i-1)} = \frac{1}{n+1} \sum_{i=2}^{n+1} \frac{1}{n-i+2} \sum_{j=i}^{n+1} F_{j:n+1}(x) .$$

We first substitute for $F_{j:n+1}(x)$ and $F_{i:i}(x)$ from equation (3.1.4). Then integrate it from 0 to $F(x)$ and use equation (3.1.3). Proceeding in a similar way r-1 times, we get

$$\sum_{i=r+1}^{n+r} \frac{1}{i(i-1) \ldots (i-r)} F_{i:i}(x)$$

$$= \frac{1}{(n+1) \ldots (n+r)} \sum_{i=r+1}^{n+r} \frac{1}{n-i+r+1} \underbrace{\sum_{s=i}^{n+r} \sum_{j=s}^{n+r} \cdots \sum_{k=m}^{n+r}}_{r \text{ summations}} F_{k:n+r}(x)$$

$$= \frac{1}{(n+1) \ldots (n+r)} \sum_{i=r+1}^{n+r} \frac{1}{n-i+r+1} \sum_{j=o}^{n+r-i} \binom{r+j-1}{j} F_{i+j:n+r}(x) .$$

$$= \frac{1}{(n+1) \ldots (n+r)} \sum_{i=r+1}^{n+r} F_{i:n+r}(x) \sum_{j=r+1}^{i} \frac{1}{n+r+1-j} \binom{r+i-j-1}{r-1} ,$$

on interchanging the order of summations.

(iv) The proof of equation (3.2.6) is similar to the proof of equation (3.2.5). Consider equation (3.2.4) which can be written as

$$\frac{1}{n+1}\sum_{i=1}^{n}\frac{1}{i}\sum_{j=o}^{i-1}F_{j+1:n+1}(x) = \sum_{i=1}^{n}\frac{F_{1:i+1}(x)}{i(i+1)} .$$

On following the steps similar to the proof of equation (3.2.5), we get equation (3.2.6).

Note that for $r = 1$, equations (3.2.5) and (3.2.6) reduce to equations (3.2.3) and (3.2.4) respectively. We now generalize these equations in another manner in the following theorem.

THEOREM 3.2.2: Following the notations of Balakrishnan and Malik (1985), let

$$c_m = \begin{cases} 1 & \text{if} \quad m = 1 \\ (n+1)\ \ldots\ (n+m-1) & \text{if} \quad m \geq 2 , \end{cases}$$

for $m = 1,2,\ldots.$ Then for an arbitrary cdf $F(x)$ and $m = 1,2,\ldots,$

(i) $$\frac{1}{n+1}\sum_{i=2}^{n+1}F_{i:n+1}(x)\sum_{j=2}^{i}\frac{1}{(n-j+2)\ \ldots\ (n-j+m+1)}$$

$$= \frac{1}{c_m}\sum_{i=2}^{n+1}\binom{i+m-3}{m-1}\frac{F_{i:i}(x)}{i(i-1)} , \qquad (3.2.9)$$

(ii) $$\frac{1}{n+1}\sum_{i=1}^{n}F_{i:n+1}(x)\sum_{j=i}^{n}\frac{1}{j(j+1)\ \ldots\ (j+m-1)}$$

$$= \frac{1}{c_m}\sum_{i=2}^{n+1}\binom{i+m-3}{m-1}\frac{F_{1:i}(x)}{i(i-1)} , \qquad (3.2.10)$$

(iii) $$\frac{1}{(n+1)\ \ldots\ (n+r)}\sum_{i=r+1}^{n+r}F_{i:n+r}(x)\sum_{j=r+1}^{i}\binom{r+i-j-1}{r-1}$$

$$\frac{1}{(n+r-j+1)\ \ldots\ (n+r-j+m)}$$

$$= \frac{1}{c_m}\sum_{i=r+1}^{n+r}\binom{i+m-r-2}{m-1}\frac{F_{i:i}(x)}{i(i-1)\ \ldots\ (i-r)} \quad \text{for } r = 1,2,\ldots,$$

(3.2.11)

(iv) $$\frac{1}{(n+1)\ \ldots\ (n+r)} \sum_{i=1}^{n} F_{i:n+r}(x) \sum_{j=i}^{n} \frac{1}{j(j+1)\ \ldots\ (j+m-1)}$$

$$\cdot \binom{j+r-i-1}{r-1} = \frac{1}{c_m} \sum_{i=r+1}^{n+r} \binom{i+m-r-2}{m-1} \frac{F_{1:i}(x)}{i(i-1)\ \ldots\ (i-r)}$$

$$\text{for } r = 1,2,\ldots \tag{3.2.12}$$

Proof: (i) Balakrishnan and Malik (1985) have shown that for $m = 1,2,\ldots,$

$$\sum_{i=1}^{n} \frac{F_{i:n}(x)}{(n-i+1)\ \ldots\ (n-i+m)} = \frac{1}{c_m} \sum_{i=1}^{n} \binom{i+m-2}{m-1} \frac{F_{i:i}(x)}{i}. \tag{3.2.13}$$

We first substitute for $F_{i:n}(x)$ and $F_{i:i}(x)$ from equation (3.1.4), and then integrate it from 0 to $F(x)$. Next on using equation (3.1.3) and interchanging the order of summations, we have

$$\frac{1}{n+1} \sum_{i=2}^{n+1} F_{i:n+1}(x) \sum_{j=2}^{i} \frac{1}{(n-j+2)\ \ldots\ (n-j+m+1)} = \frac{1}{c_m} \sum_{i=2}^{n+1} \frac{F_{i:i}(x)}{i(i-1)} \binom{i+m-3}{m-1}$$

which proves equation (3.2.9).

(ii) We start with the identity given by Balakrishnan and Malik (1985), which is

$$\sum_{i=1}^{n} \frac{F_{i:n}(x)}{i(i+1)\ \ldots\ (i+m-1)} = \frac{1}{c_m} \sum_{i=1}^{n} \binom{i+m-2}{m-1} \frac{F_{1:i}(x)}{i}. \tag{3.2.14}$$

Substituting for $F_{i:n}(x)$ and $F_{1:i}(x)$ from equation (3.1.4), it gives

$$\sum_{i=1}^{n} \frac{1}{i(i+1)\ \ldots\ (i+m-1)} \{1 - \sum_{j=o}^{i-1} \binom{n}{j} x^j (1-x)^{n-j}\}$$

$$= \frac{1}{c_m} \sum_{i=1}^{n} \binom{i+m-2}{m-1} \{\frac{1-(1-x)^i}{i}\}. \tag{3.2.15}$$

Note on substituting $x = \infty$ in equation (3.2.14), we get a combinatorial identity

$$\sum_{i=1}^{n} \frac{1}{i(i+1)\ \ldots\ (i+m-1)} = \frac{1}{c_m} \sum_{i=1}^{n} \binom{i+m-2}{m-1} \frac{1}{i} \ . \qquad (3.2.16)$$

On substracting equation (3.2.16) from (3.2.15), we have

$$\sum_{i=1}^{n} \frac{1}{i(i+1)\ \ldots\ (i+m-1)} \sum_{j=o}^{i-1} \binom{n}{j} x^j (1-x)^{n-j}$$

$$= \frac{1}{c_m} \sum_{i=1}^{n} \binom{i+m-2}{m-1} \frac{(1-x)^i}{i} \ ,$$

which on interchanging the order of summations, becomes

$$\sum_{i=o}^{n-1} \binom{n}{i} x^i \ (1-x)^{n-i} \sum_{j=i}^{n-1} \frac{1}{(j+1)\ \ldots\ (j+m)}$$

$$= \frac{1}{c_m} \sum_{i=1}^{n} \binom{i+m-2}{m-1} \frac{(1-x)^i}{i} \ .$$

Integrating it from 0 to $F(x)$ and using equation (3.1.3), we get equation (3.2.10).

(iii) The proof is identical to the proof of part (iii) of preceding theorem. Starting with equation (3.2.9) and proceeding in a similar manner r-1 times, we get

$$\frac{1}{c_m} \sum_{i=r+1}^{n+r} \binom{i+m-r-2}{m-1} \frac{F_{i:i}(x)}{i(i-1)\ \ldots\ (i-r)}$$

$$= \frac{1}{(n+1)\ \ldots\ (n+r)} \sum_{i=r+1}^{n+r} \frac{1}{(n-i+r+1)\ \ldots\ (n-i+m+r)}$$

$$\cdot \sum_{j=i}^{n+r} \sum_{k=j}^{n+r} \ldots \sum_{l=s}^{n+r} F_{l:n+r}(x)$$

$$= \frac{1}{(n+1)\ \ldots\ (n+r)} \sum_{i=r+1}^{n+r} \frac{1}{(n-i+r+1)\ \ldots\ (n-i+m+r)} \sum_{j=i}^{n+r} \binom{r+j-i-1}{j-i} F_{j:n+r}(x$$

On interchanging the order of summations, it gives equation

(iv) The proof of this is similar to the proof of (iii). We start with equation (3.2.10) which can be written as

$$\frac{1}{n+1}\sum_{i=1}^{n}\frac{1}{i(i+1)\ldots(i+m-1)}\sum_{j=1}^{i}F_{j:n+1}(x) = \frac{1}{c_m}\sum_{i=1}^{n}\binom{i+m-2}{m-1}\frac{F_{1:i+1}(x)}{i(i+1)}.$$

After (r-1) similar steps used in obtaining equation (3.2.10), we have

$$\frac{1}{c_m}\sum_{i=1}^{n}\binom{i+m-2}{m-1}\frac{F_{1:i+r}(x)}{i(i+1)\ldots(i+r)}$$

$$= \frac{1}{(n+1)\ldots(n+r)}\sum_{i=1}^{n}\frac{1}{i(i+1)\ldots(i+m-1)}\sum_{j=1}^{i}\sum_{k=1}^{j}\ldots\sum_{q=1}^{l'}F_{q:n+1}(x)$$

$$= \frac{1}{(n+1)\ldots(n+r)}\sum_{i=1}^{n}\frac{1}{i(i+1)\ldots(i+m-1)}\sum_{j=1}^{i}\binom{r+i-j-1}{r-1}F_{j:n+r}(x),$$

which on some simplification gives equation (3.2.12).

APPLICATIONS: We apply these results to the moments of distributions with cdf's $F_1(x)$, $F_2(x)$ and the exponential distribution, which give some combinatorial identities.

(i) Identity given in equation (3.2.3) is applied to the moments of order statistics from the distribution with cdf $F_1(x)$, which yields

$$\sum_{r=1}^{n}\frac{1}{r}\sum_{j=2}^{r+1}\frac{1}{n-j+2} = \sum_{r=1}^{n}\frac{1}{r^2}.$$

This has been also obtained by Joshi (1973) by using equation (3.2.8) for the exponential distribution.

(ii) For the exponential distribution given at (1.1.1) with $\sigma = 1$, it is well known that $\nu_{r:n} = \sum_{i=1}^{r}\frac{1}{n-i+1}$.

Equation (3.2.4) when applied to the moments of order statistics from the exponential distribution gives

$$\frac{1}{n+1} \sum_{i=1}^{n} \frac{1}{n-i+2} \sum_{j=1}^{n-i+1} \frac{j}{j+i-1} = \frac{n}{n+1} - \sum_{i=2}^{n+1} \frac{1}{i^2} .$$

(iii) The identity given in equation (3.2.5) when applied to the kth ($k \leq r$) order moments of order statistics given in equation (3.2.1) becomes

$$\frac{1}{(n+1) \ldots (n+r)} \sum_{i=r+1}^{n+r} \frac{(n+r) \ldots (n+r-k+1)}{(i-1) \ldots (i-k)}$$

$$\cdot \sum_{s=r+1}^{i} \frac{1}{n-s+r+1} \binom{r+i-s-1}{r-1} = \sum_{i=r+1}^{n+r} \frac{1}{(i-1) \ldots (i-r)(i-k)} .$$

(iv) Result given in equation (3.2.9) is applied by using equation (3.2.1) for $k = 1$. This yields

$$\sum_{s=2}^{n+1} \frac{1}{s-2} \sum_{j=2}^{s} \frac{1}{(n-j+2) \ldots (n-j+m+1)} = \frac{1}{c_m} \sum_{s=2}^{n+1} \binom{s+m-3}{m-1} \frac{1}{(s-1)^2} .$$

(v) Applying the identity given at equation (3.2.12) to the rth order moments of order statistics from the distribution with cdf $F_2(x)$, this gives

$$\sum_{s=1}^{n} \frac{1}{(n+r-s) \ldots (n-s+1)} \sum_{j=s}^{n} \frac{1}{j(j+1) \ldots (j+m-1)} \binom{r-s+j-1}{r-1}$$

$$= \frac{1}{c_m} \sum_{s=r+1}^{n+r} \binom{s+m-r-2}{m-1} \frac{1}{(s-1) \ldots (s-r+1)(s-r)^2} .$$

3.2.2 Case when extremes at both ends do not have finite moments

We now prove some identities which are useful when extremes do not have finite moments.

THEOREM 3.2.3: For an arbitrary cdf $F(x)$,

(i) $$\sum_{j=1}^{n} \frac{n-j+1}{(n+1)(n+2)} F_{j+1:n+2}(x) = \sum_{j=1}^{n} \frac{1}{(j+1)(j+2)} F_{2:j+2}(x), \tag{3.2.17}$$

(ii) $$\sum_{j=1}^{n} \frac{j}{(n+1)(n+2)} F_{j+1:n+2}(x) = \sum_{j=1}^{n} \frac{1}{(j+1)(j+2)} F_{j+1:j+2}(x), \tag{3.2.18}$$

(iii) $$\sum_{j=1}^{n} \binom{n}{j} \frac{(k+j-1)!\ (n+k-j)!}{(n+2k)!} F_{k+j:n+2k}(x)$$

$$= \sum_{j=1}^{n} \frac{(k)!\ (j+k-1)!}{(j+2k)!} F_{k+1:j+2k}(x), \quad k \geq 1, \tag{3.2.19}$$

(iv) $$\sum_{j=1}^{n} \binom{n}{j-1} \frac{(k+j-1)!\ (n+k-j)!}{(n+2k)!} F_{k+j:n+2k}(x)$$

$$= \sum_{j=1}^{n} \frac{(k)!\ (j+k-1)!}{(j+2k)!} F_{j+k:j+2k}(x), \quad k \geq 1. \tag{3.2.20}$$

Proof: (i) We know that

$$\sum_{j=o}^{n} \binom{n}{j} x^j (1-x)^{n-j} = 1, \tag{3.2.21}$$

or $$\sum_{j=1}^{n} \binom{n}{j} x^j (1-x)^{n-j} = 1 - (1-x)^n,$$

or $$\sum_{j=1}^{n} \binom{n}{j} x^j (1-x)^{n-j} = \sum_{j=o}^{n-1} x(1-x)^j, \tag{3.2.22}$$

multiplying both sides by $(1-x)$, we get

$$\sum_{j=1}^{n} \binom{n}{j} x^j (1-x)^{n-j+1} = \sum_{j=o}^{n-1} x(1-x)^{j+1}.$$

Integrating it from 0 to $F(x)$ and using equation (3.1.3), we have

$$\sum_{j=1}^{n} \frac{(n-j+1)}{(n+1)(n+2)} F_{j+1:n+2}(x) = \sum_{j=1}^{n} \frac{1}{(j+1)(j+2)} F_{2:j+2}(x) .$$

(ii) Again we start with the identity given in equation (3.2.21) which gives

$$\sum_{j=o}^{n-1} \binom{n}{j} x^j (1-x)^{n-j} = 1-x^n ,$$

or

$$\sum_{j=o}^{n-1} \binom{n}{j} x^j (1-x)^{n-j} = \sum_{j=o}^{n-1} (1-x) x^j . \tag{3.2.23}$$

Now multiplying both sides by x and integrating from 0 to F(x), we get the result.

(iii) We use the same technique as in (i), but multiply both sides of equation (3.2.22) by $x^{k-1}(1-x)^k$, $k \geq 1$. Then, integrating it from 0 to F(x) and simplifying, it gives the desired result.

(iv) This is the generalization of equation (3.2.18). We multiply both sides of equation (3.2.23) by $x^k(1-x)^{k-1}$, $k \geq 1$ and proceed as before to get the result.

<u>COROLLARY 3.2.1</u>: For an arbitrary cdf F(x) and for r = 1,2,3,...

(i) $$\frac{1}{(n+1)\ ..\ (n+r+2)} \sum_{j=2}^{n+2} F_{j+r:n+r+2}(x) \sum_{i=1}^{j-1} (n-i+1)\binom{r+j-i-2}{j-i-1}$$

$$= \sum_{j=1}^{n} \frac{1}{(j+1)\ ..\ (j+r+2)} \sum_{i=2}^{j+2} \binom{r+i-3}{i-2} F_{r+i:j+r+2}(x) \tag{3.2.24}$$

and

(ii) $$\frac{1}{(n+1)\ ..\ (n+r+2)} \sum_{j=2}^{n+2} F_{j+r:n+r+2}(x) \sum_{i=1}^{j-1} j\binom{r+j-i-2}{r-1}$$

$$= \sum_{i=1}^{n} \frac{1}{(j+1)\ ..\ (j+r+2)} (F_{j+r+1:j+r+2}(x) + rF_{j+r+2:j+r+2}(x)) \qquad (3.2.25)$$

Proof: (i) Here we start with equation (3.2.17) and follow the steps used in proving the results of Section 3.2.1. That is, first substituting for $F_{r:n}(x)$ from equation (3.1.4), we have

$$\sum_{j=1}^{n} \frac{(n-j+1)}{(n+1)(n+2)} \sum_{i=j+1}^{n+2} \binom{n+2}{i} x^i (1-x)^{n-i+2}$$

$$= \sum_{j=1}^{n} \frac{1}{(j+1)(j+2)} \sum_{i=2}^{j+2} \binom{j+2}{i} x^i (1-x)^{j+2-i} .$$

Integrating it from 0 to F(x) and using equation (3.1.3), it gives

$$\sum_{j=1}^{n} \frac{(n-j+1)}{(n+1)(n+2)} \sum_{i=j+1}^{n+2} \frac{1}{n+3} F_{i+1:n+3}(x)$$

$$= \sum_{j=1}^{n} \frac{1}{(j+1)(j+2)} \sum_{i=2}^{j+2} \frac{1}{j+3} F_{i+1:j+3}(x) .$$

After r similar steps, we get

$$\frac{1}{(n+1)\ ..\ (n+r+2)} \sum_{j=1}^{n} (n-j+1) \sum_{i=j+1}^{n+2} \sum_{k=i+1}^{n+3} \cdots \sum_{l=m+1}^{n+r+1} F_{l+1:n+r+2}(x)$$

$$= \sum_{j=1}^{n} \frac{1}{(j+1)\ ..\ (j+r+2)} \sum_{i=2}^{j+2} \sum_{k=i+1}^{j+3} \cdots \sum_{l=m+1}^{j+r+1} F_{l+1:j+r+2}(x) ,$$

which after some simplification reduces to equation (3.2.24).

(ii) The proof is analogous to the proof of (i) with starting equation as (3.2.18).

The results given in Corollary 3.2.1 may be applied to the moments of order statistics from the distributions for which first (r+1) moments do not exist.

<u>APPLICATIONS</u>: We apply the results of this section to the moments of order statistics from distribution with cdf $F_3(x) = \frac{1}{2}(F_1(x)+F_2(x))$, where $F_1(x)$ and $F_2(x)$ are given in Section 3.2.1. Now

$$F_3(x) = \begin{cases} -\frac{1}{2x} & -\infty < x \leq -1 \\ \frac{1}{2} & -1 < x < 1 \\ 1-\frac{1}{2x} & 1 \leq x . \end{cases}$$

For this distribution, we first evaluate $\nu_{r:n}$ for $2 \leq r \leq n-1$. Clearly

$$\nu_{r:n} = \int_{-\infty}^{\infty} x\, f_{r:n}(x)\, dx$$

$$= \frac{(n)!}{(r-1)!(n-r)!} \left[\int_{-\infty}^{-1} x\left(-\frac{1}{2x}\right)^{r-1}\left(1+\frac{1}{2x}\right)^{n-r}\frac{1}{2x^2}\, dx + \int_{1}^{\infty} x\left(1-\frac{1}{2x}\right)^{r-1}\left(\frac{1}{2x}\right)^{n-r}\frac{1}{2x^2}\, dx\right].$$

Substituting $u = -1/2x$ in first integral and $v = 1-1/2x$ in second integral, we get

$$\nu_{r:n} = \frac{(n)!}{2(r-1)!(n-r)!}\left[\int_{1/2}^{1} v^{r-1}(1-v)^{n-r-1}\, dv - \int_{0}^{1/2} u^{r-2}(1-u)^{n-r}\, du\right]$$

$$= \frac{(n)!}{2(r-1)!(n-r)!}\left[\int_{0}^{1} u^{r-1}(1-u)^{n-r-1}\, du - \int_{0}^{1/2} u^{r-1}(1-u)^{n-r-1}\, du - \int_{0}^{1/2} u^{r-2}(1-u)^{n-r}\, du\right].$$

Note that the integrals appearing in $\mu_{r:n}$ are convergent for $2 \le r \le n-1$. Using beta functions and equation (1.2.1), it gives

$$\nu_{r:n} = \frac{n}{2(n-r)} - \frac{n(n-1)}{2(r-1)(n-r)} \sum_{i=r-1}^{n-2} \binom{n-2}{i} \left(\frac{1}{2}\right)^{n-2} . \qquad (3.2.26)$$

(i) Applying the identity given at equation (3.2.17) to the moments given in equation (3.2.26), it yields

$$\sum_{j=1}^{n} \frac{1}{j} \sum_{i=j}^{n} \binom{n}{i} = 2^n - 2^{n+1} \sum_{j=1}^{n} \frac{1}{j2^{j+1}} + 2^n \sum_{j=1}^{n-1} \frac{1}{j+1}$$

$$= 2^n \sum_{j=1}^{n} \frac{1}{j} - 2^n \sum_{j=1}^{n} \frac{1}{j\ 2^j} . \qquad (A)$$

(ii) Similarly applying the result given in equation (3.2.18) to the moments given in equation (3.2.26), this gives

$$\sum_{j=1}^{n} \frac{1}{n-j+1} \sum_{i=j}^{n} \binom{n}{i} = 2^n \sum_{j=1}^{n} \frac{1}{j\ 2^j} . \qquad (B)$$

Note that on adding equations (A) and (B), we get the obvious identity $\sum_{j=1}^{n} \frac{1}{j} \sum_{k=o}^{n} \binom{n}{k} = 2^n \sum_{j=1}^{n} \frac{1}{j}$.

(iii) When the identity given at equation (3.2.17) is applied to the moments of order statistics from exponential distribution, it gives

$$\sum_{j=1}^{n} \frac{n-j+1}{(n+1)(n+2)} \sum_{i=1}^{j+1} \frac{1}{n+2-i+1} = \frac{1}{2^2} - \frac{1}{(n+2)^2} .$$

All the results given in this chapter are also true for discrete case as well. It can be proved by using the arguments similar to the Balakrishnan (1986).

CHAPTER 4

ESTIMATION OF SCALE PARAMETER OF AN EXPONENTIAL DISTRIBUTION IN A SINGLE OUTLIER EXCHANGEABLE MODEL

4.1 Introduction

Consider the exchangeable model for a single outlier from the exponential distribution given at equation (1.1.2) with random variables $X_1,\ldots,X_n$. In this and subsequent chapters, we suppress the dependence on n and denote the order statistic $X_{r:n}$ by $X_{(r)}$ only. Thus $X_{(1)} \leq X_{(2)} \cdots \leq X_{(n)}$ denote the order statistics obtained from $X_1,\ldots,X_n$.

Several authors have investigated the problem of estimating the parameter σ, while treating α as a nuisance parameter. Kale and Sinha (1971) gave an estimator of σ for $\alpha < 1$, which is a linear combination of first m order statistics, and is given by

$$U_7 = \frac{1}{(m+1)} \left[\sum_{i=1}^{m-1} X_{(i)} + (n-m+1)X_{(m)} \right], \qquad m \leq n.$$

Justification of ignoring $X_{(m+1)},\ldots,X_{(n)}$ is that the largest order statistic $X_{(n)}$ has the maximum probability of being the outlying observation when $\alpha < 1$. A similar argument holds for $X_{(m+1)},\ldots,X_{(n-1)}$ and the set $\{X_{(m+1)},\ldots,X_{(n)}\}$ has the maximum probability of containing the outlying observation.

Joshi (1972) extended this result. He calculated the optimum value m* of m for different values of n = 2(1)10(5) 20(10)50 and α = .05(.05)1.00. He has also derived an expression for the mean square error of U_7, which is given by

$$\mathrm{mse}(U_7)/\sigma^2 = \frac{1}{(m+1)} + \frac{2\theta^2}{(m+1)^2}\left[1 - (n-m)p_m\left(\frac{1}{\alpha} + \sum_{i=1}^{m} \frac{1}{n-i+\alpha}\right)\right],$$

$$\text{where} \quad \theta = \frac{(1-\alpha)}{\alpha} \quad \text{and} \quad p_m = \frac{\alpha\,\Gamma(n)\,\Gamma(n-m+\alpha)}{\Gamma(n+\alpha)\,\Gamma(n+1-m)}. \qquad (4.1.1)$$

For a given α, the estimator of Joshi denoted by U_9 is then the estimator U_7 with m replaced by m^*.

Chikkagoudar and Kunchur (1980) proposed an alternative estimator U_3 which is more efficient than U_9 in some cases. Their estimator U_3 is given by

$$U_3 = \sum_{i=1}^{n} \left(1 - \frac{2i}{n(n+1)}\right) \frac{X_{(i)}}{n}.$$

Kimber (1983) studied the robustness of estimators of σ for exponential distribution as a special case of gamma distribution. He included in his study the estimators which are linear combinations of first (n-m) observations, and are given by

$$U_6 = \sum_{i=1}^{n-m} \frac{X_{(i)}}{b(1,n)}$$

$$\text{and} \quad U_{14} = \frac{\sum_{i=1}^{n-m} X_{(i)} + m\,X_{(n-m)}}{n-m}, \quad m \leq n,$$

where $b(1,n) = E[\sum_{i=1}^{n-m} X_{(i)} | \alpha = 1)$ is the unbiasing factor. Based on some exact calculations and some simulation studies, he also concluded that U_6 is preferable to U_{14}.

In this chapter, we consider some more estimators of σ treating α (>0) as an nuisance parameter, and compare them on the basis of bias and mean square error criterion. The ml equations are solved for this case in Section 4.2 and the mle of σ is obtained. In Section 4.3, we derive exact expressions

of biases and mse's. The limiting values of biases and mse's as $\alpha \to 0$ and $\alpha \to \infty$, are calculated in Section 4.4. In Section 4.5, the two estimators which are linear combinations of two and three optimum order statistics are studied. Finally, we evaluate the biases and mse's of all the estimators considered for various values of α, and study the robustness of various estimators.

4.2 Maximum likelihood estimation

From equation (1.1.2), we get the likelihood function of the sample $x_1,\ldots,x_n$ as

$$L(\sigma,\alpha|x_1,\ldots,x_n) = \frac{\alpha}{n\sigma^n} e^{-\frac{n\bar{x}}{\sigma}} \sum_{i=1}^{n} e^{(1-\alpha)x_i/\sigma}, \quad \sigma>0,\alpha>0, \tag{4.2.1}$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$. Joshi (1988) has obtained the ml equations as

$$n\bar{x} = (n - 1 + \frac{1}{\alpha})\sigma, \tag{4.2.2}$$

$$\sum_{i=1}^{n} (x_i - \frac{\sigma}{\alpha}) e^{(1-\alpha)x_i/\sigma} = 0. \tag{4.2.3}$$

The mle of σ and α can be obtained by solving these equations. Clearly one solution of these equations is $\hat{\alpha} = 1$ and $\hat{\sigma} = \bar{x}$. In order to solve these equations for σ and α, we first eliminate α by substituting for α from equation (4.2.2) in equation (4.2.3). This gives an equation

$$\sum_{i=1}^{n} (x_i - n\bar{x} + (n-1)\sigma) e^{\frac{x_i}{\sigma}(1 - \frac{1}{n\bar{x}/\sigma - n + 1})} = 0. \tag{4.2.4}$$

We now have the following lemma.

LEMMA 4.2.1: All roots of equation (4.2.4) lie in the interval (I_1, I_2), where $I_1 = \frac{n\bar{x}-x_{(n)}}{n-1}$ and $I_2 = \frac{n\bar{x}-x_{(1)}}{n-1}$.

Proof: Let

$$h(\sigma) = \sum_{i=1}^{n} (x_i - n\bar{x} + (n-1)\sigma) e^{\frac{x_i}{\sigma}(1 - \frac{1}{n\bar{x}/\sigma - n+1})} .$$

Clearly $h(\bar{x}) = 0$ and hence $\bar{x}$ is a root of equation $h(\sigma) = 0$.

Now $h(\sigma) < 0$ if

$x_i - n\bar{x} + (n-1)\sigma < 0 \ \forall \ i.$

That is $x_{(n)} - n\bar{x} + (n-1)\sigma < 0,$

or $\sigma < \frac{n\bar{x} - x_{(n)}}{n-1} = I_1 .$

Similarly $h(\sigma) > 0$ if

$x_i - n\bar{x} + (n-1)\sigma > 0 \ \forall \ i.$

That is $x_{(1)} - n\bar{x} + (n-1)\sigma > 0,$

or $\sigma > \frac{n\bar{x} - x_{(1)}}{n-1} = I_2.$

Thus $h(\sigma)$ can be equal to 0 only in the interval (I_1, I_2).

We have not been able to obtain the mle completely due to complicated nature of $h(\sigma)$. However, the following results provide partial answers, and help in finding the mle.

RESULT (A): For $\Sigma x_i^2 < (2n-1)\bar{x}^2$, $h(\sigma)$ is an increasing function of σ at $\bar{x}$.

RESULT (B): For $\Sigma x_i^2 > (2n-1)\bar{x}^2$, $h(\sigma)$ is a decreasing function of σ at $\bar{x}$.

<u>Proof</u>: To prove Results (A) and (B), note that

$$h'(\sigma) = -\sum_{i=1}^{n} (x_i - n\bar{x} + (n-1)\sigma) e^{\frac{x_i}{\sigma}\left(1 - \frac{1}{n\bar{x}/\sigma - (n-1)}\right)} \left\{\frac{x_i}{\sigma^2} + \frac{x_i(n-1)}{(n\bar{x} - (n-1)\sigma)^2}\right\}$$

$$+ (n-1) \sum_{i=1}^{n} e^{\frac{x_i}{\sigma}\left(1 - \frac{1}{n\bar{x}/\sigma - (n-1)}\right)} .$$

This gives

$$h'(\bar{x}) = n(n-1) - \sum_{i=1}^{n}(x_i - \bar{x})\left[\frac{x_i}{\bar{x}^2} + \frac{x_i(n-1)}{\bar{x}^2}\right]$$

$$= n(2n-1) - \frac{n}{\bar{x}^2} \sum_{i=1}^{n} x_i^2 .$$

Hence if $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, then

$h'(\bar{x}) > 0$,

i.e., $h(\sigma)$ is an increasing function of σ at $\sigma = \bar{x}$. And if $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$, then $h'(\bar{x}) < 0$, i.e., $h(\sigma)$ is a decreasing function of σ at $\sigma = \bar{x}$. This completes the proof of Results (A) and (B).

<u>THEOREM 4.2.1</u>: For $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, the likelihood function has a local maximum at $(\sigma,\alpha) = (\bar{x},1)$.

<u>Proof</u>: Using equation (4.2.1), we have the logarithm of likelihood function as

$$\log L(\sigma,\alpha) = \log\alpha - \log n - n \log\sigma - \frac{n\bar{x}}{\sigma} + \log \sum_{i=1}^{n} e^{(1-\alpha)x_i/\sigma} .$$

Direct differentiation gives the second partial derivatives of $\log L(\sigma,\alpha)$ as

$$\frac{\partial^2}{\partial\alpha^2}\log L(\sigma,\alpha) = -\frac{1}{\alpha^2} + \frac{\sum_{i=1}^{n}\frac{x_i^2}{\sigma^2}e^{(1-\alpha)x_i/\sigma}}{(\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma})} - \frac{(\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma}\frac{x_i}{\sigma})^2}{(\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma})^2},$$

$$\frac{\partial^2}{\partial\sigma\partial\alpha}\log L(\sigma,\alpha) = \frac{\partial}{\partial\sigma}\left[\frac{1}{\alpha} + \frac{\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma}(-\frac{x_i}{\sigma})}{\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma}}\right]$$

$$= \frac{\sum_{i=1}^{n}\frac{x_i}{\sigma^2}e^{(1-\alpha)x_i\ \sigma} + \sum_{i=1}^{n}\frac{x_i}{\sigma}e^{(1-\alpha)x_i/\sigma}\frac{(1-\alpha)x_i}{\sigma^2}}{(\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma})}$$

$$- \frac{(1-\alpha)[\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma}\frac{x_i}{\sigma}]^2}{\sigma(\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma})^2},$$

$$\frac{\partial^2}{\partial\sigma^2}\log L(\sigma,\alpha) = \frac{\partial}{\partial\sigma}\left[\frac{n\bar{x}}{\sigma^2} - \frac{\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma}(1-\alpha)\frac{x_i}{\sigma^2}}{\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma}} - \frac{n}{\sigma}\right]$$

$$= \frac{n}{\sigma^2} - \frac{2n\bar{x}}{\sigma^3} - \frac{(1-\alpha)}{\sigma^2}\frac{\partial}{\partial\sigma}\frac{\sum_{i=1}^{n}x_i e^{(1-\alpha)x_i/\sigma}}{\sum_{i=1}^{n}e^{(1-\alpha)x_i/\sigma}}.$$

At the point $(\sigma,\alpha) = (\bar{x},1)$, these reduces to

$$\left.\frac{\partial^2}{\partial\alpha^2}\log L(\sigma,\alpha)\right|_{(\bar{x},1)} = \frac{1}{n^2\bar{x}^2}\left[n\sum_{i=1}^{n}x_i^2 - (n\bar{x})^2\right] - 1$$

$$= \frac{1}{n\bar{x}^2}\sum_{i=1}^{n}x_i^2 - 2,$$

$$\left.\frac{\partial^2}{\partial\sigma\partial\alpha}\log L(\sigma,\alpha)\right|_{(\bar{x},1)} = \frac{1}{\bar{x}}$$

$$\frac{\partial^2}{\partial \sigma^2} \log L(\sigma,\alpha)\Big|_{(\bar{x},1)} = -\frac{n}{\bar{x}^2} .$$

Let $\underset{\sim}{H}$ be a matrix formed by second partial derivatives of $\log L(\sigma,\alpha)$ at point $(\bar{x},1)$. Then $\underset{\sim}{H}$ simplifies to

$$\underset{\sim}{H} = \begin{bmatrix} \frac{1}{n\bar{x}^2} \sum_{i=1}^{n} x_i^2 - 2 & \frac{1}{\bar{x}} \\ \frac{1}{\bar{x}} & -\frac{n}{\bar{x}^2} \end{bmatrix}$$

and $$|\underset{\sim}{H}| = \frac{1}{\bar{x}^4} \left((2n-1)\bar{x}^2 - \sum_{i=1}^{n} x_i^2\right). \qquad (4.2.5)$$

Now to show that $\underset{\sim}{H}$ is a negative definite matrix, it is sufficient to show that $|\underset{\sim}{H}| > 0$ and $H_{11} < 0$. From equation (4.2.5), it is clear that if $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, then $|\underset{\sim}{H}| > 0$ and

$$H_{11} = \frac{1}{n\bar{x}^2} \sum_{i=1}^{n} x_i^2 - 2 < \frac{(2n-1)}{n} - 2 = -\frac{1}{n} ,$$

that is $H_{11} < 0$. Therefore, $\underset{\sim}{H}$ is a negative definite matrix under the conditions of the theorem. This gives that $\log L(\sigma,\alpha)$ has a local maximum at $(\sigma,\alpha) = (\bar{x},1)$. This completes the proof of Theorem 4.2.1.

<u>REMARK 1</u>: These results do not guarantee that there is no other root of equation $h(\sigma) = 0$ in the interval (I_1, I_2) in this case. However, numerical calculations show that $\bar{x}$ is indeed the only root of $h(\sigma) = 0$, and hence is the mle of σ.

<u>REMARK 2</u>: For Case (B), $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$, and $|\underset{\sim}{H}| < 0$. Consequently nothing much can be inferred in this case, as the

matrix $\underset{\sim}{H}$ is indefinite. It can be shown that along the plane given in equation (4.2.2), $L(\sigma,\alpha)$ has a local minimum at $(\bar{x},1)$ in this case. Next for the case $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$, equation $h(\sigma) = 0$ has at least two roots other than $\bar{x} = \sigma_2$ (say). Let these roots be σ_1, σ_3. Then $\sigma_1 \in (I_1,\bar{x})$ and $\sigma_3 \in (\bar{x},I_2)$. This can be seen easily by using Lemma 4.2.1.

We have proved that if $\sigma \in (-\infty,I_1)$, then $h(\sigma)$ is negative and $h(\sigma)$ is a decreasing function at $\bar{x}$ with $h(\bar{x}) = 0$. This implies that there will be at least one root σ_1 in the interval $(I_1,\bar{x})$. In a similar manner, it can be shown that there will be at least one root σ_3 in the interval $(\bar{x},I_2)$. This is illustrated in Figure 4.2.1.

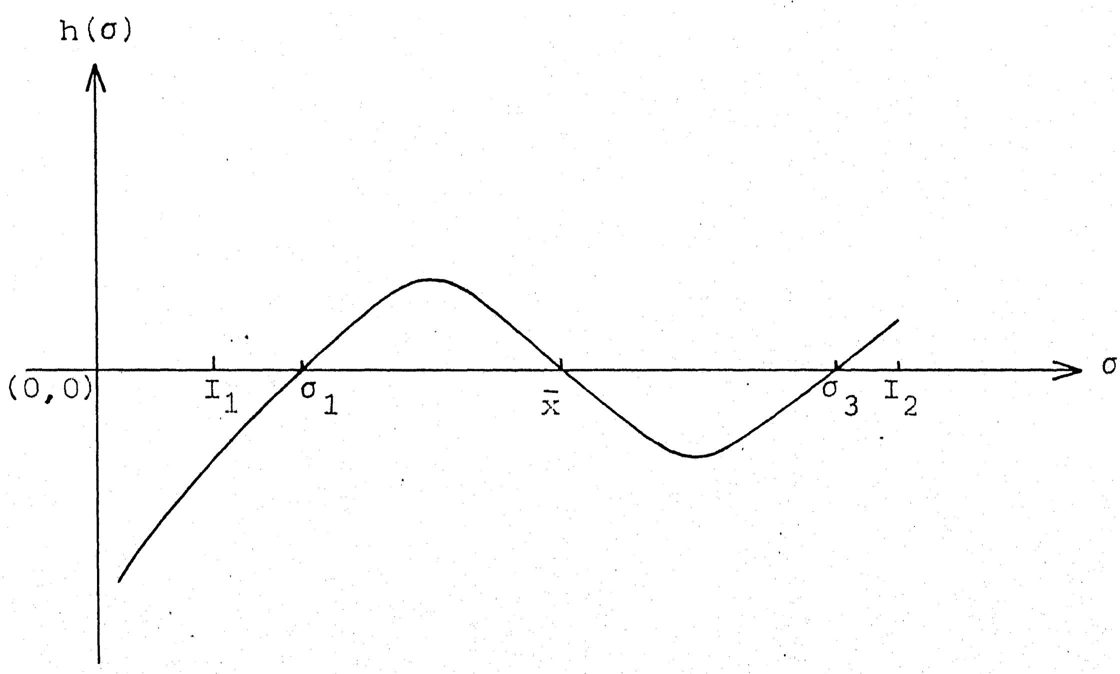

Figure 4.2.1. Showing the graph of $h(\sigma)$ when $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$.

Therefore, for Case (B), mle will be that root out of σ_1, σ_2, σ_3 which maximizes the likelihood function. In this case, our numerical calculations show that there are only two roots namely σ_1, σ_3 other than $\sigma_2 = \bar{x}$ of equation $h(\sigma) = 0$, and also reveal that mle of σ is either σ_1 or σ_3.

A computer program for finding the mle is given in Appendix B. On solving ml equations [(4.2.2) and (4.2.3)], it is observed that $\hat{\alpha}$ may also be greater than 1. This can be observed by the following example.

EXAMPLE 1: Ten observations were generated from the exchangeable single outlier model given at equation (1.1.2) for $\sigma = 1$ and $\alpha = 0.1$. The sample values are given in Table 4.2.1 under data set (I). This sample gave the ml estimate $\hat{\alpha} = 3.0648$ and $\hat{\sigma} = 1.7960$. Some of these sample values were subsequently changed to get other data sets. These are summarized in Table 4.2.2 along with $L_i = L(\sigma_i, \alpha_i/x_1,\ldots,x_n)$. This table also gives points I_1, $\bar{x}$ and I_2 so that $\sigma_1 \in (I_1,\bar{x})$ and $\sigma_3 \in (\bar{x},I_2)$.

It can be seen that the mle of α is very sensitive to even minor changes in observations, while σ is not much affected by such changes. This is also expected since α is essentially based on one observation only. For example, only one observation is different in data sets(III) and (IV), but α changes from 1.0000 to 0.6139. In data sets(II) and (IV) there is some difference in 3 observations, but for one case mle of σ is greater than $\bar{x}$, while for other case it is less than $\bar{x}$. For the data set (III), the condition $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$ is satisfied so that there is only one root $\bar{x}$.

TABLE 4.2.1: Data sets from exchangeable single outlier model

Data set no.	Data sets arranged in increasing order of magnitude	$\sum_{i=1}^{n} x_i^2-(2n-1)\bar{x}^2$
(I)	.18, .30, .49, .55, .79, 1.35, 1.54, 1.94, 3.53, 6.08	5.3672
(II)	.30, .49, .55, .67, .79, 1.35, 1.54, 1.94, 3.53, 5.59	.0654
(III)	.30, .49, .55, .68, .79, 1.35, 1.54, 1.94, 2.53, 5.50	-.3266
(IV)	.30, .49, .55, .68, .79, 1.35, 1.54, 1.94, 2.53, 5.70	.6649

TABLE 4.2.2: Showing mle of σ and α and points I_1, $\bar{x}$, I_2

Data set no.	I_1	$\bar{x}$	I_2	i	σ_i	α_i	$10^6 L_i$	mle σ	mle α
I	1.186	1.675	1.841	1	1.479	0.4301	0.2684		
				2	1.675	1.0000	0.2612		
				3	1.796	3.0648	0.2761	1.796	3.0648
II	1.240	1.675	1.828	1	1.671	0.9766	0.2612		
				2	1.675	1.0000	0.2612		
				3	1.733	1.5011	0.2613	1.733	1.5011
III	1.130	1.567	1.708	2	1.567	1.0000	0.5086	1.567	1.0000
IV	1.130	1.587	1.730	1	1.493	0.6139	0.4489		
				2	1.587	1.0000	0.4480		
				3	1.622	1.2752	0.4482	1.493	0.6139

4.3 Different estimators of σ and their biases and mean square errors

Due to the difficulty in calculating the mle, we now restrict our attention to the estimators which are linear combinations of order statistics such as winsorized mean, trimmed mean etc. We study following estimators as they have been used by others in similar situations, see for example, David and Shu (1978), Chikkagoudar and Kunchur (1980), Balakrishnan and Ambagaspitiya (1988) etc. Some others are also included because of their intuitive appeal. The estimators are as follows:

(i) Sample mean:

$$U_1 = \sum_{i=1}^{n} X_i/n.$$

(ii) Minimum mean square error estimator in the homogeneous case:

$$U_2 = \sum_{i=1}^{n} X_i/(n+1).$$

(iii) Chikkagoudar and Kunchur's (1980) estimator:

$$U_3 = \sum_{i=1}^{n} \left(1 - \frac{2i}{n(n+1)}\right) \frac{X_{(i)}}{n}.$$

(iv) One-sided trimmed mean with largest observation deleted:

$$U_4 = \frac{1}{(n-1)} \sum_{i=1}^{n-1} X_{(i)},$$ which is suggested by Kale (1975).

(v) Limiting minimum mean square error estimator:

$$U_5 = \frac{1}{n} \sum_{i=1}^{n-1} X_{(i)}.$$

(vi) Estimator suggested by Kimber (1983):

$$U_6 = \frac{1}{b(1,n)} \sum_{i=1}^{n-1} X_{(i)}, \text{ where } b(1,n) = \sum_{i=1}^{n-1} \sum_{j=1}^{i} \frac{1}{n-j+1}.$$

(vii) One-sided winsorized mean of Kale and Sinha (1971):

$$U_7 = \frac{1}{n} [\sum_{i=1}^{n-1} X_{(i)} + X_{(n-1)}].$$

(viii) One-sided trimmed mean with two largest observations deleted:

$$U_8 = \frac{1}{(n-2)} \sum_{i=1}^{n-2} X_{(i)}.$$

(ix) Joshi's (1972) estimator:

$$U_9 = \frac{1}{m^* + 1} [\sum_{i=1}^{m^*} X_{(i)} + (n-m^*) X_{(m^*)}],$$

where $m^* \leq n$, is the optimum value of m defined in Section 4.1.

(x) Winsorized mean with one observation missing from each end:

$$U_{10} = \frac{1}{n} [2(X_{(2)} + X_{(n-1)}) + \sum_{i=3}^{n-2} X_{(i)}].$$

(xi) Symmetrically trimmed mean with one observation missing from each end:

$$U_{11} = \frac{1}{(n-2)} \sum_{i=2}^{n-1} X_{(i)}.$$

(xii) Winsorized mean with two observations missing from each end:

$$U_{12} = \frac{1}{n} [3(X_{(3)} + X_{(n-2)}) + \sum_{i=4}^{n-3} X_{(i)}].$$

(xiii) Symmetrically trimmed mean with two observations missing from each end:

$$U_{13} = \frac{1}{(n-4)} \sum_{i=3}^{n-2} X_{(i)} .$$

In most of the estimators considered, we are deleting either some largest observations or some largest and smallest observations. Justification for the non-inclusion of such observations is that for $\alpha < 1$, the set of some largest observations has the maximum probability of containing the outlying observation. But for $\alpha > 1$, the smallest observation has the maximum probability of being the outlier. Similar arguments hold for $X_{(2)},\ldots,X_{(i)}$. We have also included U_2 which is the minimum mean square error estimator in homogeneous case and U_1 which is the uniformly minimum variance unbiased estimator in the homogeneous case.

We now evaluate exact expressions of biases and mse's of above estimators. These expressions will be used in next sections for comparing various estimators. For evaluating these expressions, we write each estimator as $U = \sum_{i=1}^{n} d_i Z_i$, by using the transformation given by

$$Z_r = (n-r+1)(X_{(r)}-X_{(r-1)}), \quad r = 1,\ldots,n ,$$

where $X_{(0)} = 0$, then $X_{(i)} = \sum_{k=1}^{i} \frac{Z_k}{n-k+1}$. (4.3.1)

Here wlog we assume that $\sigma = 1$. This transformation is studied by Joshi (1972) for this model. Some results of this paper, which will be used, are stated below. Let $\theta = (1-\alpha)/\alpha$ and

$$p_r = \text{probability that rth order statistic is an outlier}$$
$$= \frac{\alpha\,\Gamma(n)\,\Gamma(n-r+\alpha)}{\Gamma(n+\alpha)\,\Gamma(n-r+1)}$$
$$= \frac{\alpha}{nb_1\cdots b_r}, \qquad (4.3.2)$$

where $b_j = \frac{(n-j+\alpha)}{(n-j+1)}$, $j = 1,\ldots,n.$

For $i = 1,\ldots,n,$

(i) $$\sum_{r=i}^{n} p_r = \frac{(n-i+\alpha)}{\alpha}\, p_i,$$

$$\sum_{r=i}^{n} rp_r = \frac{(n-i+\alpha)(n+i\alpha)}{\alpha(\alpha+1)}\, p_i, \qquad (4.3.3)$$

(ii) $$E(Z_i) = 1 + \theta p_i,$$

$$E(Z_i^2) = 2 + 2\theta p_i\,\frac{(2n-2i+1+\alpha)}{(n-i+\alpha)},$$

$$E(Z_iZ_j) = 1 + \theta p_i + \theta p_j\,\frac{(n-i+1)}{(n-i+\alpha)} \quad \text{for } 1 \le i < j \le n. \qquad (4.3.4)$$

The transformed estimators are then as follows:

(i) $$U_1 = \sum_{i=1}^{n} Z_i/n\ ;$$

(ii) $$U_2 = \sum_{i=1}^{n} Z_i/(n+1)\ ;$$

(iii) $$U_3 = \frac{1}{(n+1)} \sum_{i=1}^{n} \left(1 - \frac{i}{n^2}\right) Z_i\ ;$$

(iv) $$U_4 = \frac{1}{(n-1)} \sum_{i=1}^{n-1} \frac{(n-i)}{(n-i+1)}\, Z_i\ ;$$

(v) $$U_5 = \frac{1}{n} \sum_{i=1}^{n-1} \frac{(n-i)}{(n-i+1)}\, Z_i\ ;$$

(vi) $U_6 = \frac{1}{b(1,n)} \sum_{i=1}^{n-1} \frac{(n-i)}{(n-i+1)} Z_i$;

(vii) $U_7 = \frac{1}{n} \sum_{i=1}^{n-1} Z_i$;

(viii) $U_8 = \frac{1}{(n-2)} \sum_{i=1}^{n-2} (\frac{n-i-1}{n-i+1}) Z_i$;

(ix) $U_9 = \frac{1}{(m^*+1)} \sum_{i=1}^{m^*} Z_i$;

(x) $U_{10} = \frac{1}{n} [Z_1 + \frac{n}{(n-1)} Z_2 + Z_3 + \ldots Z_{n-1}]$;

(xi) $U_{11} = \frac{Z_1}{n} + \frac{1}{(n-2)} \sum_{i=2}^{n-1} \frac{(n-i)}{(n-i+1)} Z_i$;

(xii) $U_{12} = \frac{1}{n} [Z_1 + \frac{n}{(n-1)} Z_2 + \frac{n}{(n-2)} Z_3 + Z_4 + \ldots + Z_{n-2}]$;

(xiii) $U_{13} = \frac{Z_1}{n} + \frac{Z_2}{(n-1)} + \frac{1}{(n-4)} \sum_{i=3}^{n-2} \frac{(n-i-1)}{(n-i+1)} Z_i$.

All these estimators are of the form $U = \sum_{i=1}^{n} d_i Z_i = \underset{\sim}{d}'\underset{\sim}{Z}$ (say), where $\underset{\sim}{d}' = (d_1, \ldots, d_n)$. This gives

$$E(U) = \sum_{i=1}^{n} d_i \, E(Z_i),$$

$$E(U^2) = \underset{\sim}{d}' E(\underset{\sim}{Z}\, \underset{\sim}{Z}')\, \underset{\sim}{d},$$

$$\text{mse}(U) = E(U-1)^2 = E(U^2) - 2E(U) + 1,$$

which can be obtained by using equations (4.3.3) and (4.3.4). For simplifying some of these expressions such as for U_4, U_8, U_{11}, U_{13}, etc., we need the following lemma.

<u>LEMMA 4.3.1</u>: For $m = 1,\dots,n$,

$$\sum_{i=1}^{m} \frac{p_i}{(n-i+1)} = \begin{cases} \dfrac{p_m}{1-\alpha} - \dfrac{\alpha}{n(1-\alpha)} & \text{for } \alpha \neq 1 \\[2ex] \dfrac{1}{n} \displaystyle\sum_{i=1}^{m} \frac{1}{n-i+1} & \text{for } \alpha = 1. \end{cases} \tag{4.3.5}$$

<u>Proof</u>: Using equation (4.3.2), we get

$$\sum_{i=1}^{m} \frac{p_i}{(n-i+1)} = \alpha \sum_{i=1}^{m} \frac{\Gamma(n)\,\Gamma(n-i+\alpha)}{\Gamma(n+\alpha)\,\Gamma(n-i+2)} .$$

For $\alpha = 1$, it gives $\sum_{i=1}^{m} p_i/(n-i+1) = (1/n) \sum_{i=1}^{m} 1/(n-i+1)$. For $\alpha \neq 1$, we apply a result proved by Vännman (1976), which states that

$$\sum_{i=1}^{m} \frac{\Gamma(d-i)}{\Gamma(e-i+1)} = \begin{cases} \displaystyle\sum_{i=1}^{m} \frac{1}{(e-i)} & \text{if } d = e \\[2ex] \dfrac{1}{(d-e)} \left[\dfrac{\Gamma(d)}{\Gamma(e)} - \dfrac{\Gamma(d-m)}{\Gamma(e-m)}\right] & \text{if } d \neq e, \end{cases} \tag{4.3.6}$$

where m (≥ 1) is an integer and d and e are real or complex numbers such that all gamma functions involved are well defined. This gives

$$\sum_{i=1}^{m} \frac{p_i}{(n-i+1)} = \frac{\alpha\,\Gamma(n)}{\Gamma(n+\alpha)} \frac{1}{(\alpha-1)} \left(\frac{\Gamma(n+\alpha)}{\Gamma(n+1)} - \frac{\Gamma(n+\alpha-m)}{\Gamma(n+1-m)}\right)$$

$$= \frac{\alpha}{(\alpha-1)n} - \frac{\alpha}{(\alpha-1)} \frac{\Gamma(n)\,\Gamma(n+\alpha-m)}{\Gamma(n+\alpha)\,\Gamma(n+1-m)}$$

$$= \frac{p_m}{(1-\alpha)} - \frac{\alpha}{n(1-\alpha)} .$$

This completes the proof of Lemma 4.3.1.

Using this technique, exact expressions for mse's of U_1, U_2, U_3 have been obtained. It can be easily shown that

$$E(U_1) = \frac{1}{n}\left(n - 1 + \frac{1}{\alpha}\right),$$

$$E(U_1^2) = \frac{1}{n^2}\ (n - 1 + \frac{1}{\alpha^2}) + \frac{1}{n^2}\ (n - 1 + \frac{1}{\alpha})^2 \quad \text{and}$$

$$\text{mse}(U_1) = \frac{1}{n} + \frac{2\theta}{n^2\alpha} . \qquad (4.3.7)$$

Joshi (1972) obtained the mse of U_2 as

$$\text{mse}(U_2) = \frac{1}{(n+1)} + \frac{2\theta^2}{(n+1)^2} . \qquad (4.3.8)$$

The expression for mse of U_3 is given by Chikkagoudar and Kunchur (1980).

Next, we compare U_1 and U_2 on the basis of mse criterion by using exact expressions of mse's. From equations (4.3.7) and (4.3.8), we get

$$\text{mse}(U_2) - \text{mse}(U_1)$$

$$= [\frac{\alpha^2 n(n-1) - 2\alpha(n^2-2n-1) - 2(2n+1)}{\alpha^2 n^2 (n+1)^2}].$$

Since the term in denominator is positive, consider the term given in numerator. The positive root of the equation $n(n-1)\alpha^2 - 2(n^2-2n-1)\alpha - 2(2n+1) \geq 0$

is $\quad \alpha_1 = \dfrac{n^2-2n-1 + \sqrt{(n^2-2n-1)^2 + 2(2n+1)(n^2-n)}}{n(n-1)}.$

This gives that if α is greater than α_1, mse(U_2) will be large than mse(U_1). It can be seen that as $n \to \infty$, α_1 tends to 2. For some smaller values of n, α_1 is given in the Table 4.3.1. It shows that for $\alpha \geq 2$, U_1 is better than U_2 for all n.

TABLE 4.3.1: Showing the values of α_1 for n = 5, 10, 20 and 100

n	α_1
5	1.9610
10	1.9901
20	1.9975
100	1.9990

To illustrate the use of Lemma 4.3.1, we obtain the bias and mse of the estimator $U' = c \sum_{i=1}^{n-1} X_{(i)}$ which corresponds to U_4, U_5 and U_6 for different choices of c. Now

$$\text{Bias}(U') = E(c \sum_{i=1}^{n-1} \frac{n-i}{n-i+1} Z_i) - 1.$$

Using equation (4.3.4), we get

$$\text{Bias}(U') = c[\, n-1 - \sum_{i=1}^{n-1} \frac{1}{n-i+1}] + c\theta[1 - p_n - \sum_{i=1}^{n-1} \frac{p_i}{n-i+1}] - 1.$$

On applying Lemma 4.3.1 and equation (4.3.2), it gives

$$\text{Bias}(U') = c((n-1) - \sum_{i=1}^{n-1} \frac{1}{n-i+1}) + c\theta - c\theta p_n - c\theta(\frac{p_{n-1}}{1-\alpha} - \frac{\alpha}{n(1-\alpha)}) - 1$$

$$= c(n-1) + c\theta + \frac{c}{n} - 1 - c \sum_{i=1}^{n-1} \frac{1}{n-i+1} - c\theta p_n - \frac{c\theta\alpha p_n}{(1-\alpha)}$$

$$= c(n-1) + c\theta + \frac{c}{n} - 1 - c \sum_{i=1}^{n-1} \frac{1}{n-i+1} - \frac{cp_n}{\alpha} .$$

Similarly mse of U' is given by

$$\text{mse}(U') = E(c \sum_{i=1}^{n-1} X_i - 1)^2$$

$$= E(c(n\bar{X} - X_{(n)}) - 1)^2$$

$$= c^2n^2E(\bar{X}^2)+c^2E(X^2_{(n)})-2ncE(\bar{X}X_{(n)})+1-2ncE(\bar{X})+2cE(X_{(n)}).$$

For evaluating this, we need some expressions which are given below and are easy to establish.

$$E(X_{(n)}) = \frac{p_n}{\alpha} + \sum_{i=2}^{n} \frac{1}{n-i+1} .$$

Gross <u>et al</u>. (1986) have obtained $E(X^2_{(n)})$ as

$$E(X^2_{(n)}) = \sum_{j=1}^{n-1} \frac{1}{j^2} + 2B(\alpha,n) \sum_{j=o}^{n-1} \frac{1}{j+\alpha} + (\sum_{j=1}^{n-1} \frac{1}{j})^2 .$$

$E(\bar{X})$ and $E(\bar{X}^2)$ are given in equation (4.3.7). Finally,

$$E(\bar{X}\ X_{(n)}) = E(\frac{1}{n} \sum_{i=1}^{n} Z_i \sum_{j=1}^{n} \frac{Z_j}{n-j+1})$$

$$= \frac{1}{n}\ E(\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{Z_iZ_j}{(n-j+1)})$$

$$= \frac{1}{n}\ (\sum_{i=1}^{n} \frac{E(Z_i^2)}{n-i+1} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \frac{E(Z_iZ_j)}{n-j+1}).$$

The last term can be evaluated by using equations (4.3.3) and (4.3.4). Substituting these values we can evaluate mse(U'). Similar expression of mse(U') is found if we follow the method described previously.

In the latter study, U_5 and U_6 are not included. These estimators are similar to U_4. A brief table for biases and mse's of U_4, U_5 and U_6 for n = 10, 20 is given in Table 4.3.2.

The table values reveal that mse(U_4) is less than mse(U_6) for all tabulated values of n and α. The mse(U_5) is less than mse(U_4) for some very small values of α only. It is

well known that for $\alpha = 1$ (homogeneous case), U_2 is best and in general the estimators which use all n observations perform better for large α values than these estimators (Chikkagoudar and Kunchur, 1980). Hence, we have considered only U_4 out of U_4, U_5, U_6 for smaller α values.

4.4 Limiting values of biases and mean square errors of various estimators

Exact expressions for mse of some estimators described in Section 4.3, are rather complicated and it is not easy to compare them directly. In this section, we evaluate the biases and mse's of various estimators for three cases, viz., (a) $\alpha \to 0$, (b) $\alpha = 1$ and (c) $\alpha \to \infty$. The estimator U_9 is not considered in this section because m* is not well defined. These limiting values give a rough idea about the relative performance of various estimators. We discuss these cases one by one. Without loss of generality, we take $\sigma = 1$ throughout this section.

Case (a): Limits as $\alpha \to 0$. Equation (4.3.2) can be rewritten as

$$p_i = \frac{\alpha\,\Gamma(n)}{\Gamma(n-i+1)}\,\frac{1}{(n+\alpha-1)\,\ldots\,(n+\alpha-i)}\,.$$

This form of p_i immediately gives

$$\lim_{\alpha\to 0} p_i = \begin{cases} 0 & \text{for } i = 1,2,\ldots,n-1 \\ 1 & \text{for } i = n\,, \end{cases}$$

$$\lim_{\alpha\to 0} \frac{p_i}{\alpha} = \begin{cases} 1/(n-i) & \text{for } i = 1,\ldots,n-1 \\ \infty & \text{for } i = n\,, \end{cases}$$

$$\lim_{\alpha\to 0} \frac{\theta p_i}{\alpha} = \infty \quad \text{for } i = 1,\ldots,n. \tag{4.4.1}$$

Applying equations (4.3.4) and (4.4.1), we get

$$\lim_{\alpha\to 0} E(Z_i) = \begin{cases} 1+1/(n-i) & \text{for } i = 1,\ldots,n-1 \\ \infty & \text{for } i = n\,, \end{cases}$$

$$\lim_{\alpha\to 0} E(Z_i^2) = \begin{cases} 2 + \dfrac{2(2n-2i+1)}{(n-i)^2} & \text{for } i = 1,\ldots,n-1 \\ \infty & \text{for } i = n\,, \end{cases}$$

$$\lim_{\alpha\to 0} E(Z_iZ_j) = \begin{cases} 1 + \dfrac{1}{(n-i)} + \dfrac{1}{(n-j)}\,\dfrac{(n-i+1)}{(n-i)} & \text{for } 1\le i<j\le n-1 \\ \infty & \text{for } i=1,\ldots,n-1, j=n. \end{cases}$$

(4.4.2)

We illustrate the calculation of limiting values of bias and mse for U_{10} which is given by

$$U_{10} = \frac{1}{n}\left(Z_1 + \frac{n}{(n-1)}Z_2 + Z_3 + \ldots + Z_{n-1}\right). \tag{4.4.3}$$

On using equation (4.4.2), we have

$$\lim_{\alpha\to 0} E(U_{10}) = \frac{1}{n}\left[\left(1 + \frac{1}{n-1}\right) + \frac{n}{n-1}\left(1 + \frac{1}{n-2}\right) + \sum_{i=3}^{n-1}\left(1 + \frac{1}{n-i}\right)\right]$$

$$= \frac{n-3}{n} + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\sum_{i=3}^{n-1}\frac{1}{n-i}\,.$$

This gives the limiting bias $B(U_{10}) = E(U_{10}-1)$ as

$$\lim_{\alpha\to 0} B(U_{10}) = -\frac{3}{n} + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\sum_{i=3}^{n-1}\frac{1}{n-i}\,.$$

For calculating the limiting value of mse, we first evaluate the limit of $E(U_{10}^2)$. Now U_{10} can be written as

$$U_{10} = \frac{1}{n}(\underset{\sim}{d}_1' - \underset{\sim}{d}_2')\,\underset{\sim}{Z}_{(n)}\,,$$

where $\underset{1\times(n-1)}{\underset{\sim}{d}_1'} = (1,\ldots,1)$,

$$\underset{1X(n-1)}{\underset{\sim}{d}_2'} = (0, -\frac{1}{(n-1)}, 0, \ldots, 0)$$

and $$\underset{1X(n-1)}{\underset{\sim}{Z}_{(n)}'} = (Z_1, \ldots, Z_{n-1}).$$

$$E(U_{10}^2) = \frac{1}{n^2}[\underset{\sim}{d}_1' E(\underset{\sim}{Z}_{(n)}\underset{\sim}{Z}_{(n)}')\underset{\sim}{d}_1 - 2\underset{\sim}{d}_1' E(\underset{\sim}{Z}_{(n)}\underset{\sim}{Z}_{(n)}')\underset{\sim}{d}_2$$
$$+ \underset{\sim}{d}_2' E(\underset{\sim}{Z}_{(n)}\underset{\sim}{Z}_{(n)}')\underset{\sim}{d}_2]. \tag{4.4.4}$$

On using equation (4.4.2), we get

$$\lim_{\alpha\to 0} \underset{\sim}{d}_1' E(\underset{\sim}{Z}_{(n)}\underset{\sim}{Z}_{(n)}')\underset{\sim}{d}_1$$

$$= (1,\ldots,1)\begin{bmatrix} 2+\frac{2(2n-1)}{(n-1)^2} & \frac{n}{n-1}(1+\frac{1}{n-2}) & \cdots & \frac{n}{n-1}(1+\frac{1}{1}) \\ \frac{n}{n-1}(1+\frac{1}{n-2}) & 2+\frac{2(2n-3)}{(n-2)^2} & & \frac{n-1}{n-2}(1+\frac{1}{1}) \\ \vdots & & & \vdots \\ \frac{n}{n-1}(1+\frac{1}{1}) & \cdots\cdots\cdots & & 2+\frac{2\cdot 3}{1^2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$= (1,\ldots,1)\begin{bmatrix} 2+\frac{2(2n-1)}{(n-1)^2}+\frac{n}{(n-1)}\sum_{i=2}^{n-1}(\frac{n-i+1}{n-i}) \\ 2+\frac{2(2n-3)}{(n-2)^2}+\frac{(n-1)}{(n-2)}\sum_{i=1}^{n-1}(\frac{n-i+1}{n-i})-(\frac{(n-1)}{(n-2)})^2 \\ 2+\frac{2(2n-3)}{(n-3)^2}+\frac{(n-2)}{(n-3)}\sum_{i=1}^{n-1}(\frac{n-i+1}{n-i})-(\frac{n-2}{n-3})^2 \\ \vdots \\ 2+\frac{2\cdot 3}{1^2}+\frac{2}{1}\sum_{i=1}^{n-1}(\frac{n-i+1}{n-i})-(\frac{2}{1})^2 \end{bmatrix}.$$

$$= 2(n-1) + 2\sum_{i=1}^{n-1}\left(\frac{2n-2i+1}{(n-i)^2}\right) + \left(\sum_{i=1}^{n-1}\frac{n-i+1}{n-i}\right)^2 - \sum_{i=1}^{n-1}\left(\frac{n-i+1}{n-i}\right)^2 .$$

Similarly on using equation (4.4.2), we get

$$\lim_{\alpha\to 0} \underset{\sim}{d}_2' \, E(\underset{\sim}{Z}_{(n)}\underset{\sim}{Z}_{(n)}') \, \underset{\sim}{d}_2 = \frac{1}{(n-1)^2}\left(2 + \frac{2(2n-3)}{(n-2)^2}\right)$$

and

$$\lim_{\alpha\to 0} \underset{\sim}{d}_1' \, E(\underset{\sim}{Z}_{(n)}\underset{\sim}{Z}_{(n)}')\underset{\sim}{d}_2 = -\frac{1}{(n-1)}\left[\frac{n}{(n-2)} + 2 + \frac{2(2n-3)}{(n-2)^2} + \left(\frac{n-1}{n-2}\right)\sum_{i=3}^{n-1}\frac{n-i+1}{n-i}\right].$$

Substituting these values in equation (4.4.4), we get

$$\lim_{\alpha\to 0} E(U_{10}^2) = \frac{1}{n^2}\Bigg[2(n-1) + 2\sum_{i=1}^{n-1}\frac{(2n-2i+1)}{(n-i)^2} + \left(\sum_{i=1}^{n-1}\frac{n-i+1}{n-i}\right)^2 -$$

$$\sum_{i=1}^{n-1}\left(\frac{n-i+1}{n-i}\right)^2 + \frac{2n}{(n-1)(n-2)} + \frac{4}{(n-1)} + \frac{4(2n-3)}{(n-1)(n-2)^2} +$$

$$\frac{2}{(n-2)}\sum_{i=3}^{n-1}\frac{n-i+1}{n-i} + \frac{1}{(n-1)^2}\left\{2 + \frac{2(2n-3)}{(n-2)^2}\right\}\Bigg].$$

Now

$$\mathrm{mse}(U_{10}) = E(U_{10}^2) + 1 - 2E(U_{10}).$$

On taking limit as $\alpha \to 0$ and substituting the values of $\lim_{\alpha\to 0} E(U_{10}^2)$ and $\lim_{\alpha\to 0} E(U_{10})$, we get

$$\lim_{\alpha\to 0} \mathrm{mse}(U_{10}) = \frac{1}{n^2}\Bigg[2(n-1) + 4\sum_{i=1}^{n-1}\frac{1}{n-i} + 2\sum_{i=1}^{n-1}\frac{1}{(n-i)^2} +$$

$$\left(\sum_{i=1}^{n-1}\frac{n-i+1}{n-i}\right)^2 - \sum_{i=1}^{n-1}\left(\frac{n-i+1}{n-i}\right)^2 + \frac{2n}{n-2} - \frac{2(n-2)}{n-1} +$$

$$\frac{2(2n-3)(2n-1)}{(n-1)^2(n-2)^2} + \frac{2(n-3)}{n-2} + \frac{2}{n-2}\sum_{i=3}^{n-1}\frac{1}{n-i} + \frac{2}{(n-1)^2}\Bigg] +$$

$$1 - 2\left[\frac{n-3}{n} + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\sum_{i=3}^{n-1}\frac{1}{n-i}\right].$$

The expressions for limiting biases and mse's of other estimators can be calculated in a similar manner. Note that, for estimators U_1, U_2, U_3, which contain $X_{(n)}$, both the limiting bias and mse are infinite. Finite bias and mse expressions are tabulated in Table 4.4.1.

Case (b): $\alpha = 1$. In this homogeneous case, most expressions simplify considerably. Now $\theta = 0$ and $p_i = \frac{1}{n}$ for all i. Further $Z_1, Z_2, \ldots, Z_n$ are i.i.d. exponential variates with common pdf $f(z) = \exp(-z)$. In this case, U_2 has the least mse equal to $\frac{1}{(n+1)}$. Expressions for biases and mse's of all estimators are tabulated in Table 4.4.2.

Case (c): Limits as $\alpha \to \infty$. In this case, small observations indicate the presence of outliers. Hence this case loses its importance since such observations are bounded below by zero. However, we still obtain the limiting biases and mse's. Now

$$\lim_{\alpha \to \infty} p_i = \begin{cases} 1 & \text{for } i = 1 \\ 0 & \text{for } i = 2,3,\ldots,n , \end{cases}$$

$$\lim_{\alpha \to \infty} \frac{p_i}{\alpha} = 0 \quad \text{for } i = 1,\ldots,n,$$

and

$$\lim_{\alpha \to \infty} \theta\, p_i = \begin{cases} 0 & \text{for } i = 2,\ldots,n \\ -1 & \text{for } i = 1 . \end{cases} \tag{4.4.5}$$

Applying equations (4.3.4) and (4.4.5), we get

$$\lim_{\alpha \to \infty} E(Z_i) = \begin{cases} 0 & \text{for } i = 1 \\ 1 & \text{for } i = 2,\ldots n , \end{cases}$$

$$\lim_{\alpha \to \infty} E(Z_i^2) = \begin{cases} 0 & \text{for } i = 1 \\ 2 & \text{for } i = 2,\ldots,n , \end{cases}$$

and

$$\lim_{\alpha\to\infty} E(Z_i Z_j) = \begin{cases} 0 & \text{for } i = 1,\ j = 2,\ldots,n \\ 1 & \text{for } 2 \le i < j \le n\ . \end{cases} \qquad (4.4.6)$$

We illustrate this method also by evaluating bias and mse of U_{10} which is given in equation (4.4.3). On using equation (4.4.6), we get

$$\lim_{\alpha\to\infty} E(U_{10}) = \frac{1}{n}\left(\frac{n}{(n-1)} + 1 + \ldots + 1\right)$$

$$= \frac{1}{(n-1)} + \frac{(n-3)}{n}\ .$$

This gives bias $B(U_{10})$ as

$$\lim_{\alpha\to\infty} B(U_{10}) = \frac{1}{(n-1)} - \frac{3}{n}\ .$$

In a similar manner, for calculating the limiting mse, we evaluate $E(U_{10}^2)$. We can write U_{10} as

$$U_{10} = \frac{1}{n} \underset{1Xn-1}{\underset{\sim}{d}'} \ \underset{n-1X1}{\underset{\sim}{Z}}\ ,$$

where $\underset{1Xn-1}{\underset{\sim}{d}'} = (1, \frac{n}{n-1}, 1,\ldots,1)$

and $\underset{1Xn-1}{\underset{\sim}{Z}'} = (Z_1, Z_2,\ldots,Z_{n-1})$.

Using these expressions, we have

$$\lim_{\alpha\to\infty} E(U_{10}^2) = \lim_{\alpha\to\infty} \frac{1}{n^2} \underset{\sim}{d}' E(\underset{\sim}{Z}\,\underset{\sim}{Z}') \underset{\sim}{d}\ .$$

On using equation (4.4.6), it gives

$$\lim_{\alpha\to\infty} E(U_{10}^2) = \frac{2}{(n-1)^2} + \frac{2(n-3)}{n(n-1)} + \frac{(n-3)(n-2)}{n^2}\ .$$

Consequently

$$\lim_{\alpha \to \infty} \text{mse}(U_{10}) = \frac{n^3 - 5n + 6}{n^2(n-1)^2}$$

Expressions for biases and mse's of all estimators are given in Table 4.4.3.

These calculations show that for small values of α, the estimator containing the largest observation is undesirable. But if α is large, U_1 performs better than U_2, U_3, U_7, U_{10}, U_{12}. We have tabulated these limiting values of biases and mse's of these estimators for n = 10 in Table 4.4.4. These values show that for small values of α, U_4 is better than all other estimators considered in this section. This conclusion remains unchanged for other values of n as well. Next best estimator is U_8 for $n < 10$ and U_{13} for $n \geq 10$. However, U_{11} is not far behind than U_{13} for large values of n. For $\alpha = 1$, U_2 is the best for all values of n. The next best is U_3. For large values of α $(\to \infty)$, U_1 is the best for all values of n. The next best is U_2. It is thus clear that different estimators perform better for different values of α and no single estimator can be used for all cases.

4.5 Estimators based on few optimum order statistics

The use of few optimum order statistics for the estimation of parameters has received considerable attention in literature, for example, see David (1981, Section 7.6) for a summary description. This technique is also applicable when we have a type II censored sample (Saleh, 1966). Some other such estimators using only a selected few order statistics are Gastwirth type estimators (Gastwirth and Cohen, 1970). In

general, such estimators are robust and the outlying observations have very little effect on these estimators.

We now extend the table given by Saleh (1966) for right censored samples, which is appropriate for our present case. Briefly speaking, the procedure of Saleh for exponential distribution is as follows.

For a given k, consider the order statistics $X_{(n_1)},\ldots,X_{(n_k)}$ with ranks $n_1,\ldots,n_k$, determined by k fixed real numbers $p_1^*,\ldots,p_k^*$ satisfying the order relation $0 < p_1^* < \ldots < p_k^* < 1$ and $n_i = [np_i^*] + 1$, $i = 1,\ldots,k$. Define $p_0^* = 0$, $p_{k+1}^* = 1$. To estimate the scale parameter σ; let $t_i = -\log(1-p_i^*)$, $i = 1,\ldots,k$, where $0 < t_1 < \ldots < t_k \leq -\log(1-\beta)$ and $1-\beta$ is the proportion of censoring on the right. Set $t_k = -\log(1-\beta)$. This gives the rank of corresponding order statistic as $n_k = [n\beta] + 1$. Thus the largest available observation in the relevant sample is chosen. The remaining (k-1) order statistics are the solution of system of equations

$$\tau_{i+1} + \tau_i - 2t_i = 0, \qquad i = 1,\ldots,(k-1), \tag{4.5.1}$$

where τ_i is given by the equation

$$1 - \tau_i = \frac{t_{i-1}e^{-t_{i-1}} - t_i e^{-t_i}}{e^{-t_i} - e^{-t_{i-1}}} .$$

Finally,

$$\hat{\sigma} = \sum_{i=1}^{k} b_i^* x_{(n_i)} , \tag{4.5.2}$$

where expressions for b_i^*'s are given in Saleh's paper.

For the exchangeable single outlier model with $\alpha < 1$, it is reasonable to delete $X_{(n)}$ and take $n_k = n-1$. This gives

$p_k^* = \frac{n-2}{n} = 1 - \frac{2}{n}$ and $t_k = -\log(\frac{2}{n})$. To determine the remaining $(k-1)$ optimum order statistics, we solve equations (4.5.1) for $t_1, t_2, \ldots, t_{k-1}$.

Here we consider only two values of k, viz., $k = 2, 3$. We have solved the system of equations given in equation (4.5.1) for $k = 2,3$ by using Newton-Raphson method. When t_i's are known, we evaluate p_i^* by setting

$$p_i^* = 1 - e^{-t_i}, \quad i = 1,\ldots,(k-1).$$

Thus the ranks of the remaining $(k-1)$ optimum order statistics are given by

$$n_i = [np_i^*] + 1, \quad i = 1,\ldots,(k-1).$$

Now estimate of σ is given at equation (4.5.2). We denote the two estimates by $\sigma^{(2)}$, $\sigma^{(3)}$ for $k = 2$ and 3 respectively. To determine b_i's we follow a slightly different procedure and minimize the mse $(\hat{\sigma})$ with respect to $b_1,\ldots,b_k$ in the homogeneous case $(\alpha = 1)$. For $k = 2$, this gives b_1, b_2 as the solution of the equations

$$b_1 E(X^2_{(n_1)}) + b_2 E(X_{(n_1)} X_{(n_2)}) = E(X_{(n_1)}),$$

$$b_1 E(X_{(n_1)} X_{(n_2)}) + b_2 E(X^2_{(n_2)}) = E(X_{(n_2)}). \qquad (4.5.3)$$

For $k = 3$, b_1, b_2 and b_3 are the solution of the equations

$$b_1 E(X^2_{(n_1)}) + b_2 E(X_{(n_1)} X_{(n_2)}) + b_3 E(X_{(n_1)} X_{(n_3)}) = E(X_{(n_1)}),$$

$$b_1 E(X_{(n_1)} X_{(n_2)}) + b_2 E(X^2_{(n_2)}) + b_3 E(X_{(n_2)} X_{(n_3)}) = E(X_{(n_2)}),$$

$$b_1 E(X_{(n_1)} X_{(n_3)}) + b_2 E(X_{(n_2)} X_{(n_3)}) + b_3 E(X^2_{(n_3)}) = E(X_{(n_3)}).$$

(4.5.4)

For $n = 4(2)20$, the t_i's, p_i^*'s, n_i's and b_i's are given in Tables 4.5.1 and 4.5.2 for $k = 2$ and 3 respectively. These estimators are easy to use because these contain only 2 or 3 order statistics and are also expected to be robust. It can be seen from these tables that for $k = 2$, $n_1 \doteq [\frac{2}{3} n_2]$, $n_2 = (n-1)$ and for $k = 3$, $n_1 \doteq [\frac{n_3}{2}]$, $n_2 \doteq [\frac{3}{4} n_3]$, $n_3 = (n-1)$. Some other calculations also reveal the same pattern. With n_i's obtained approximately as above, b_i's can be obtained by solving equations (4.5.3) and (4.5.4) for the two cases. Another estimator using only some order statistics is the Gastwirth mean G given by

$$G = .3(X_{([\frac{n}{3}]+1)} + X_{(n-[\frac{n}{3}])}) + .2(X_{([\frac{n}{2}])} + X_{([\frac{n}{2}]+1)}).$$

Next we compare the estimators defined above and the estimators obtained by using equation (4.5.2) for $k = 3$ with weights b_i^* given by Saleh (1966). For a proper comparison, and use of Saleh's table, 10% censoring is taken for $n = 10$ and 5% censoring for $n = 20$ as only one observation is deleted from the sample. This estimator contains $X_{(n)}$ and is given by

$$\sigma_1^{(3)} = \begin{cases} .4156X_{(5)} + .2546X_{(8)} + .1821X_{(10)}, & \text{for } n = 10 \\ .4373X_{(11)} + .2354X_{(17)} + .1029X_{(20)}, & \text{for } n = 20 . \end{cases}$$

Another comparable set of estimator for $k = 3$ which uses $X_{(n-1)}$ is obtained by 20% and 10% censoring for $n = 10$ and 20 respectively. This is given by

$$\sigma_2^{(3)} = \begin{cases} .3921X_{(4)} + .2763X_{(7)} + .3459X_{(9)}, & \text{for } n = 10 \\ .4156X_{(9)} + .2546X_{(15)} + .1821X_{(19)}, & \text{for } n = 20 . \end{cases}$$

It may be noted that both the estimators $\sigma^{(3)}$ and $\sigma_2^{(3)}$ use the same order statistics for n = 10 and 20. But the corresponding weights are different. For large values of n, there is some marginal difference in the order statistics used in $\sigma^{(3)}$ and $\sigma_2^{(3)}$.

The mse of $\sigma^{(2)}$, $\sigma^{(3)}$, G, $\sigma_1^{(3)}$ and $\sigma_2^{(3)}$ are given in Table 4.5.3 which gives that the estimator $\sigma^{(3)}$ is the best. This table shows that most of the time the estimator $\sigma^{(2)}$ which contains only two order statistics is better than G, $\sigma_1^{(3)}$, $\sigma_2^{(3)}$, which use more than two order statistics. It may be noted that the relative performance of $\sigma_1^{(3)}$ and $\sigma_2^{(3)}$ for n = 10 and n = 20 is considerably different. For n = 10, we have the expected result, that is, $\sigma_2^{(3)}$ is better than $\sigma_1^{(3)}$ for smaller values of α and $\sigma_1^{(3)}$ is better than $\sigma_2^{(3)}$ for large α values. But for n = 20, $\sigma_2^{(3)}$ is better than $\sigma_1^{(3)}$ for all values of α, including α = 1. In our opinion there are two possible reasons for this discrepancy. The first is that these are asymptotically best linear unbiased estimators and hence their relative performance cannot be predicted when exact mse's are calculated for small values of n. The second reason is the limited tabulation in Saleh's (1966) table. Thus, for example a 3% or a 4% censoring for n = 20 will also require an estimator with $X_{(20)}$ along with other optimum order statistics and different weights. Thus the modified estimators $\sigma^{(2)}$ and $\sigma^{(3)}$ can be used. Note that the weights b_i's can easily be calculated by means of a simple computer program, since they require the moments of order statistics from the exponential distribution in the homogeneous case only.

4.6 Comparison of different estimators of σ using exact values

In this section, we have compared various estimators of σ on the basis of mse's. In Section 4.4, we have tabulated the limiting values of mse of estimators for n = 10 and have concluded that as $\alpha \to 0$, U_4 is better than others and U_1 performs the best among all the estimators considered as $\alpha \to \infty$. For $\alpha = 1$, i.e., homogeneous case, it is well known that U_2 has the minimum mse. So, for other values of α, the biases and mse's of various estimators are calculated by the method described in Section 4.3. Biases and mse's for n = 10 and 20 are tabulated in Table 4.6.1 and Table 4.6.2 for various α values and $\sigma = 1$. The estimators considered in this comparative study, include the estimators given in Sections 4.3 and 4.5. A summary table listing the various estimators which perform best for different values of α in steps of 0.05 and for various values of n, is given in Table 4.6.3. These tables show that U_1, U_2, U_3, U_4 and U_{11} perform better than others in different ranges of α. For a given n, one can use Table 4.6.3 for finding the best estimator if α is known. It is observed that in general for a given n, the best estimator changes from U_4 to U_{11} to U_3 to U_2 to U_1 as α increases from 0 to ∞. However, for $\alpha \leq 0.4$, U_4 is almost as good as U_{11}. Thus for $\alpha \leq 0.4$ among the sample sizes studied ($n \leq 30$), the least efficiency of U_4 compared to U_{11} is 0.96 which occurs for n = 30 and $\alpha = 0.4$. Thus one may ignore U_{11}, and use U_4 in this range of α values. Similar comparison hold for other α values as well, where some other estimator is better in borderline cases.

Consequently, as a thumb rule we suggest to use

$$T = \begin{cases} U_4 & \text{for} \quad 0.00 < \alpha \leq 0.40 \\ U_3 & \text{for} \quad 0.40 < \alpha \leq 0.75 \\ U_2 & \text{for} \quad 0.75 < \alpha < 2.00 \\ U_1 & \text{for} \quad 2.00 \leq \alpha < \infty \end{cases} \tag{4.6.1}$$

for all values of n. This estimator is expected to perform best among all estimators in various sample sizes. It can be used if α is known. Thus the estimators which perform better in various α ranges, are the one-sided trimmed mean U_4 of Kale (1975), Chikkagoudar and Kunchur's(1980) estimator U_3, the minimum mse estimator in the homogeneous case U_2 and the sample mean U_1.

The main problem in using this estimator T, is that α is not known. So we first estimate α and then on the basis of the estimate of α, we can get an estimator of σ by using Table 4.6.3. For obtaining an estimate of α, we may follow any one of the following procedures.

(i) We can use the ml equations given in equations (4.2.2) and (4.2.3) which give mle of α as well. The corresponding estimate of σ is denoted by T(1).

(ii) Joshi (1972) has suggested an ad-hoc procedure for selecting an optimum value m* of m for the estimator U_9 defined in Section 4.3. This is an iterative process which uses a table of values of m* and equation

$$\frac{n^2}{(n+1)} \bar{x} = (n - 1 + \frac{1}{\alpha})\hat{\sigma} , \tag{4.6.2}$$

for obtaining an estimate of α. Here $\hat{\sigma}$ is the estimate

of σ obtained at the previous step with initial value as U_7 given in Section 4.3. The final estimate of σ using equation (4.6.1) is denoted by T(2). We have extended his table by considering more α values. The value of m* for n = 3(1)20(10)50 and α = .01(.01)1.00, are given in Table 4.6.4.

(iii) Substituting $\hat{\sigma} = U_7$ in the equation (4.6.2) given by Joshi (1972), we get another estimate of α, and an estimate of σ as T(3).

We have thus three methods for finding estimate of α. It is not possible to find the relative performances of various estimators T(1), T(2) and T(3) by finding exact mse's. We, therefore, carry out this comparison by simulation in next section.

It may also be noted that the estimator $\sigma^{(3)}$, which uses only three order statistics in the manner defined in Section 4.5, is also quite good for $\alpha \geq 0.5$. However, Joshi's (1972) estimator U_9 is uniformly better than $\sigma^{(3)}$ and hence $\sigma^{(3)}$ can be dropped for simulation studies and only those estimators which perform better than others may be studied.

4.7 Comparison of various estimators by simulation

In the last section, we have proposed an estimator by combining various estimators for different α values. Here we have compared mle with other estimators by simulation using 1000 iterations for n = 10, and 500 iterations for n = 20. First a random sample of size n (= 10 and 20), is generated from exponential distribution with $\sigma=1$. Then an observation is chosen randomly and its mean is made different from the mean of other observations. Hence we get a sample containing a single outlier.

The estimators included are U_1, U_2, U_3, U_4, U_9, U_{11} and proposed estimators using all three methods of evaluating α, which are T(1), T(2) and T(3). The mmle which is proposed by Joshi (1988), is also included, and is given by

$$(n-1)^2 \hat{\sigma}_4^2 = \left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2 .$$

Simulated values of biases and mse's are given in Table 4.7.1 and Table 4.7.2 respectively. It may be noted that the simulated values of mse are slightly higher than the exact mse values in most cases, wherever available. So the exact values of mse of estimator for which only simulation is done are likely to be slightly less than simulated values of mse.

With the limited simulation, it appears that mle does not perform well compared to other estimators included in this study. The mmle when compared with mle performs quite well for $\alpha \geq .5$. In fact, it is almost as good as the estimators U_1, U_2 and U_3, all of which use the complete sample and for

small α values, its mse is smaller than the mse's of these estimators. It is also observed from the tabulated values that T(2) is very marginally better than T(3). Compared to T(2) and T(3), the estimator T(1) is good only for large α values. In fact for small values of α, it is relatively bad. Also, out of these three estimators, T(3) is easiest to calculate. The mse of Joshi's estimator U_9 is higher than those of U_4, U_{11} and T(2) for smaller α values ($\alpha \leq 0.5$), while for larger values of α, its mse is greater than mse's of U_1, U_2 and U_3. U_4 and U_{11} perform well and are better for small α values. Table 4.7.1 also reveals that T(1), T(2), T(3), U_4 and U_{11} have small biases. From mse consideration, U_4 is better for small α values. But in the homogeneous case, U_4 has the larger mse compared to other. Therefore, to protect against outliers, we have to pay the price in terms of a larger mse when there is no outlier. But if one is interested in estimating σ such that the estimate is close to the true value, then T(3) is better. Its bias is less compared to others.

The conclusion of this simulation study can thus be summarized as follows: If the experimenter does not want to carry out the single outlier analysis but suspects an outlying observation with larger mean, then he may use U_4 or U_{11}. However, for the single outlier analysis and estimator with small bias, he should use T(3). The main advantage of T(3) is its simplicity. For finding T(3), first an estimate of α using equation

$$\frac{n}{n+1} \sum_{i=1}^{n} x_i = (n - 1 + \frac{1}{\alpha}) \hat{\sigma}$$

with $\hat{\sigma} = U_7$, is obtained. Then using equation (4.6.1), we get $T(3)$ depending on the estimate of α.

TABLE 4.3.2: Showing the biases and mse's of U_4, U_5 and U_6 for $\alpha = .0(.05).20$

n	α	Bias			mse		
		U_4	U_5	U_6	U_4	U_5	U_6
10	.00	-.0000	-.1000	.2728	.1111	.1000	.2544
	.05	-.0248	-.1224	.2412	.1078	.1018	.2318
	.10	-.0467	-.1420	.2134	.1056	.1040	.2131
	.15	-.0660	-.1594	.1887	.1042	.1063	.1975
	.20	-.0832	-.1749	.1669	.1035	.1088	.1843
20	.00	-.0000	-.0500	.1584	.0526	.0500	.0957
	.05	-.0173	-.0665	.1383	.0519	.0510	.0883
	.10	-.0323	-.0807	.1209	.0515	.0521	.0823
	.15	-.0454	-.0931	.1058	.0514	.0532	.0774
	.20	-.0568	-.1040	.0926	.0515	.0544	.0734

TABLE 4.4.1: Showing the limiting biases and mse's for $\alpha \to 0$

Estimator	Bias ($\alpha \to 0$)	mse ($\alpha \to 0$)
U_1	*	*
U_2	*	*
U_3	*	*
U_4	0	$\frac{1}{n-1}$
U_7	$-\frac{1}{n}+\frac{1}{n}\sum_{i=1}^{n-1}\frac{1}{i}$	$\frac{1}{n^2}[2(n-1)+\sum_{i=1}^{n-1}\frac{2(2n-2i+1)}{(n-i)^2}+(\sum_{i=1}^{n-1}\frac{n-i+1}{n-i})^2-\sum_{i=1}^{n-1}(\frac{n-i+1}{n-i})^2]+1-2[\frac{n-1}{n}+\frac{1}{n}\sum_{i=1}^{n-1}\frac{1}{i}]$
U_8	$-\frac{1}{n-2}\sum_{i=1}^{n-2}\frac{1}{n-i}$	$\frac{1}{(n-2)^2}[(n-2)(n-1)-4\sum_{i=1}^{n-2}\frac{1}{n-i+1}+2\sum_{i=1}^{n-2}\frac{1}{(n-i+1)(n-i)^2}-\sum_{i=1}^{n-3}\frac{n-2-i}{n-i}-\sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\{\frac{1}{n-j}-\frac{1}{(n-j)(n-i)}\}]+1-2(1-\frac{1}{n-2}\sum_{i=1}^{n-2}\frac{1}{n-i})$
U_{10}	$\frac{1}{n-2}+\frac{1}{n-1}+\frac{1}{n}\sum_{i=3}^{n-1}\frac{1}{n-i}-\frac{3}{n}$	$\frac{1}{n^2}[n^2+2n-2+4\sum_{i=1}^{n-1}\frac{1}{n-i}+2\sum_{i=1}^{n-1}\frac{1}{(n-i)^2}+(\sum_{i=1}^{n-1}\frac{n-i+1}{n-i})^2-\sum_{i=1}^{n-1}(\frac{n-i+1}{n-i})^2+\frac{2}{n-2}\sum_{i=2}^{n-2}\frac{1}{n-i-1}+\frac{1}{n}\sum_{i=3}^{n-1}\frac{1}{n-i}-2n^2(\frac{1}{n+1}+\frac{n-3}{n})+\frac{2n-5}{n-2}+$

TABLE 4.4.1 (continued)

Estimator	Bias ($\alpha \to 0$)	mse ($\alpha \to 0$)
		$\frac{2}{(n-1)^2}(1+\frac{2n-3}{(n-2)^2}) + \frac{2}{(n-1)(n-2)} \cdot \{3n-4+2\frac{(2n-3)}{(n-2)}\}]$
U_{11}	$\frac{1}{n-1}$	$\frac{1}{(n-2)^2}[n - \frac{2(n^2+2n-1)(2n-3)}{n^2(n-1)^2}]$
U_{12}	$\frac{1}{n-1}+\frac{1}{n-2}+\frac{1}{n-3}+\frac{1}{n}\sum_{i=2}^{n-4}\frac{1}{i}-\frac{5}{n}$	$\lim_{\alpha\to 0} mse(U_{10}) + \frac{8(n-1)}{n^2(n-2)^2}(1+\frac{2n-5}{(n-3)^2}) + \frac{4}{n^2(n-3)}(-2n+\sum_{i=4}^{n-1}\frac{1}{n-i}) + \frac{4}{n} - \frac{4(2n-3)}{n(n-1)(n-2)}\frac{n-4}{n-3} - \frac{4}{n^2}(n-4+\sum_{i=3}^{n-2}\frac{1}{n-i})$
U_{13}	$\frac{1}{n-1}+\frac{1}{n-2}-\frac{1}{n-4}\sum_{i=3}^{n-2} X \frac{1}{n-i}$	$1-\frac{4}{n-1}+\frac{2}{(n-1)^2}+\frac{2}{(n-2)^2} + \frac{1}{(n-4)^2}\sum_{i=3}^{n-2}(\frac{n-i-1}{n-i+1})^2(2+\frac{(2n-2i+1)2}{(n-i)^2}) + \frac{2}{(n-4)^2}\sum_{i=3}^{n-3}\frac{n-i-1}{n-i}\sum_{j=i+1}^{n-2}\frac{n-j-1}{n-j} + \frac{2}{n-4}(\frac{1}{n-2}-\frac{n-2}{n-1})\sum_{i=3}^{n-2}\frac{n-i-1}{n-i}$

* The biases and mse's of U_1, U_2, U_3 are not finite.

TABLE 4.4.2: Showing the biases and mse's for $\alpha = 1$

Estimator	Bias ($\alpha = 1$)	mse ($\alpha = 1$)
U_1	0	$\frac{1}{n}$
U_2	$-\frac{1}{n}$	$\frac{1}{n+1}$
U_3	$-\frac{(3n+1)}{2n(n+1)}$	$\frac{12n^3 + 3n^2 + 7n + 2}{12n^3(n+1)}$
U_4	$-\frac{1}{n-1}\sum_{i=1}^{n-1}\frac{1}{n-i+1}$	$\frac{1}{(n-1)^2}\left[\left(\sum_{i=1}^{n-1}\frac{n-i}{n-i+1}\right)^2 + \sum_{i=1}^{n-1}\left(\frac{n-i}{n-i+1}\right)^2\right]$ $+1-2+\frac{2}{n-1}\sum_{i=1}^{n-1}\frac{1}{n-i+1}$
U_7	$-\frac{1}{n}$	$\frac{1}{n}$
U_8	$-\frac{2}{n-2}\sum_{i=1}^{n-2}\frac{1}{n-i+1}$	$\frac{1}{(n-2)^2}\left[\left(\sum_{i=1}^{n-2}\frac{n-i-1}{n-i+1}\right)^2 + \sum_{i=1}^{n-2}\left(\frac{n-i-1}{n-i+1}\right)^2\right]$ $+ 1 - \frac{2}{n-2}\sum_{i=1}^{n-2}\frac{n-i-1}{n-i+1}$
U_{10}	$\frac{1}{n-1} - \frac{2}{n}$	$\frac{n^3-2n^2+n+2}{n^2(n-1)^2}$
U_{11}	$\frac{1}{n} - \frac{1}{n-2}\sum_{i=2}^{n-1}\frac{1}{n-i+1}$	$\frac{n^2-2n+2}{n^2} - \frac{2(n-1)}{n(n-2)}\sum_{i=2}^{n-1}\frac{n-i}{n-i+1} +$ $\frac{1}{(n-2)^2}\left[\left(\sum_{i=2}^{n-1}\frac{n-i}{n-i+1}\right)^2 + \sum_{i=2}^{n-1}\left(\frac{n-i}{n-i+1}\right)^2\right]$

contd...

TABLE 4.4.2 (continued)

Estimator	Bias ($\alpha = 1$)	mse ($\alpha = 1$)
U_{12}	$\frac{1}{n-1} + \frac{1}{n-2} - \frac{4}{n}$	$\frac{n^5-4n^4-5n^3+54n^2-92n+48}{n^2(n-1)^2(n-2)^2}$
U_{13}	$\frac{1}{n} + \frac{1}{n-1} - \frac{2}{n-4} \sum\limits_{i=3}^{n-2} \frac{1}{n-i+1}$	$\frac{3}{n^2} + \frac{3}{(n-1)^2} + \frac{2(3-2n)}{n(n-1)} +$ $2\left(\sum\limits_{i=3}^{n-2} \frac{(n-i-1)}{(n-i+1)(n-4)}\right)^2 + \frac{1}{(n-4)^2}$ $\cdot \sum\limits_{i=3}^{n-2} \left(\frac{n-i-1}{n-i+1}\right)^2 + 4\left(\frac{1}{n} + \frac{1}{n-1} - \frac{n-4}{2}\right)\frac{1}{n-4}$ $\cdot \sum\limits_{i=3}^{n-2} \frac{n-i-1}{(n-4)(n-i+1)} + 1$

TABLE 4.4.3: Showing the limiting biases and mse's for $\alpha \to \infty$

Estimator	Bias $(\alpha \to \infty)$	mse $(\alpha \to \infty)$
U_1	$-\frac{1}{n}$	$\frac{1}{n}$
U_2	$\frac{-2}{n+1}$	$\frac{1}{n+1} + \frac{2}{(n+1)^2}$
U_3	$-\frac{(5n^2+n-2)}{2n^2(n+1)}$	$\frac{1}{(n+1)^2}[\frac{n+1}{12n^3}(12n^3+3n^2+7n+2)+$ $\frac{n-1}{n^3}(3n^2+4n+1)]$
U_4	$-\frac{1}{n-1} - \sum_{i=2}^{n-1} \frac{1}{(n-1)(n-i+1)}$	$\frac{1}{(n-1)^2}[\sum_{i=2}^{n-1}(\frac{n-i}{n-i+1})^2 + (\sum_{i=2}^{n-1}\frac{n-i}{n-i+1})^2]+1$ $-\frac{2}{n-1}\sum_{i=2}^{n-1}\frac{n-i}{n-i+1}$
U_7	$-\frac{2}{n}$	$\frac{n+2}{n^2}$
U_8	$-\frac{1}{n-2} - \frac{2}{n-2}\sum_{i=2}^{n-2}\frac{1}{n-i+1}$	$\frac{1}{(n-2)^2}[\sum_{i=2}^{n-2}(\frac{n-i-1}{n-i+1})^2 + (\sum_{i=2}^{n-2}\frac{n-i-1}{n-i+1})^2]+$ $1-\frac{2(n-3)}{(n-2)} + \frac{4}{n-2}\sum_{i=2}^{n-2}\frac{1}{n-i+1}$
U_{10}	$\frac{1}{n-1} - \frac{3}{n}$	$\frac{n^3-5n+6}{n^2(n-1)^2}$
U_{11}	$-\frac{1}{n-2}\sum_{i=2}^{n-1}\frac{1}{n-i+1}$	$\frac{1}{(n-2)^2}[\sum_{i=1}^{n-2}\frac{1}{(n-i)^2} + (\sum_{i=1}^{n-2}\frac{1}{n-i})^2]+$ $\frac{1}{(n-2)^2} + \frac{2(n-4)}{(n-2)^2}\sum_{i=1}^{n-2}\frac{1}{n-i} - \frac{n-3}{n-2}$
U_{12}	$\frac{1}{n-1} + \frac{1}{n-2} - \frac{5}{n}$	$\frac{n^3-5n+6}{n^2(n-1)^2} + \frac{1}{n^2}[\frac{8}{(n-2)^2} + 2 - \frac{4}{n-2}]-$

contd...

TABLE 4.4.3 (continued)

Estimator	Bias $(\alpha \to \infty)$	mse $(\alpha \to \infty)$
		$-\frac{4}{n(n-2)} + \frac{2}{n} + \frac{4}{n(n-2)}\left[\frac{1}{n-1} + \frac{n-2}{n}\right] - \frac{2}{n}\left[\frac{1}{n-1} + \frac{n-2}{n}\right]$
U_{13}	$\frac{1}{n-1} - \frac{2}{n-4}\sum_{i=3}^{n-2}\frac{1}{n-i+1}$	$\left(\sum_{i=3}^{n-2}\frac{n-i-1}{(n-i+1)(n-4)} + \frac{1}{n-1}\right)^2 + \frac{1}{(n-1)^2} - \frac{n+1}{n-1} + \frac{1}{(n-4)^2}\sum_{i=3}^{n-2}\left(\frac{n-i-1}{n-i+1}\right)^2 + \frac{4}{n-4}\sum_{i=3}^{n-2}\frac{1}{n-i+1}$

TABLE 4.4.4: Showing the limiting biases and mse's of various estimators in three cases for n = 10

Estimators	Bias			mse		
	$\alpha \to 0$	$\alpha = 1$	$\alpha \to \infty$	$\alpha \to 0$	$\alpha = 1$	$\alpha \to \infty$
U_1	*	.0000	-.1000	*	.1000	.1000
U_2	*	-.0909	-.1818	*	.0909	.1074
U_3	*	-.1409	-.2309	*	.0937	.1191
U_4	.0000	-.2143	-.3143	.1111	.1162	.1591
U_7	.1829	-.1000	-.2000	.1954	.1000	.1200
U_8	.2286	-.3572	-.4572	.1286	.1821	.2535
U_{10}	.1954	-.0750	-.1884	.2031	.1002	.1180
U_{11}	.1111	-.1286	-.2286	.1485	.1028	.1286
U_{12}	.0240	-.1639	-.2639	.1329	.1148	.1409
U_{13}	-.0294	.1948	-.2948	.1212	.1225	.1615

* The biases and mse's of U_1, U_2 and U_3 are infinite.

TABLE 4.5.1: Showing the values of t_i's, p_i^*'s, n_i's and b_i's for k = 2

n	t_1	t_2	p_1^*	n_1	n_2	b_1	b_2
4	.3265	.6931	.2786	2	3	.3434	.5051
6	.4989	1.0986	.3928	3	5	.4152	.3950
8	.6127	1.3863	.4581	4	7	.4648	.3343
10	.6961	1.6094	.5013	6	9	.4392	.2709
12	.7611	1.7918	.5328	7	11	.4757	.2475
14	.8138	1.9459	.5568	8	13	.5067	.2292
16	.8578	2.0794	.5759	10	15	.4838	.2021
18	.8953	2.1972	.5915	11	17	.5105	.1915
20	.9278	2.3026	.6046	13	19	.4908	.1736

TABLE 4.5.2: Showing the values of t_i's, p_i^*'s, n_i's and b_i's for k = 3

n	t_1	t_2	p_1^*	p_2	n_1	n_2	n_3	b_1	b_2	b_3
4	.2137	.4438	.1924	.3584	1	2	3	.2500	.2500	.5000
6	.3233	.6859	.2763	.4964	2	3	5	.2387	.2865	.3907
8	.3946	.8493	.3260	.5723	3	5	7	.3079	.2650	.2937
10	.4462	.9713	.3600	.6214	4	7	9	.3552	.2482	.2362
12	.4862	1.0676	.3850	.6562	5	8	11	.3289	.2773	.2202
14	.5184	1.1468	.4045	.6824	6	10	13	.3608	.2631	.1893
16	.5452	1.2137	.4203	.7029	7	12	15	.3872	.2510	.1662
18	.5680	1.2713	.4334	.7195	8	13	17	.3650	.2750	.1611
20	.5878	1.3218	.4444	.7333	9	15	19	.3859	.2641	.1453

TABLE 4.5.3: Table for mse's of various estimators based on optimum order statistics for n = 10, 20

Estimator / α	$\sigma^{(2)}$	$\sigma^{(3)}$	G	$\sigma_1^{(3)}$	$\sigma_2^{(3)}$
n = 10					
.1	.1833	.1694	.1287	5.9731	.4857
.2	.1534	.1438	.1288	1.4243	.3926
.3	.1347	.1277	.1294	.6488	.3283
.4	.1229	.1174	.1305	.3995	.2830
.5	.1156	.1109	.1318	.2934	.2506
.6	.1110	.1069	.1333	.2401	.2269
.7	.1083	.1045	.1349	.2103	.2094
.8	.1068	.1031	.1367	.1921	.1962
.9	.1061	.1025	.1384	.1803	.1863
1.0	.1059	.1023	.1402	.1722	.1786
2.0	.1111	.1071	.1559	.1486	.1520
5.0	.1189	.1153	.1767	.1435	.1461
10.0	.1207	.1183	.1838	.1437	.1461
n = 20					
.1	.0899	.0798	.0840	1.8824	.1381
.2	.0757	.0686	.0865	.4615	.1122
.3	.0673	.0619	.0888	.2247	.0956
.4	.0623	.0579	.0910	.1501	.0847
.5	.0593	.0554	.0931	.1190	.0774
.6	.0576	.0540	.0950	.1035	.0724
.7	.0566	.0532	.0968	.0948	.0689
.8	.0561	.0527	.0984	.0895	.0664
.9	.0558	.0525	.1000	.0860	.0647
1.0	.0558	.0525	.1015	.0836	.0634
2.0	.0572	.0538	.1118	.0760	.0594
5.0	.0591	.0558	.1221	.0741	.0590
10.0	.0595	.0565	.1250	.0741	.0592

TABLE 4.6.1: Exact values of biases for some estimators

Est. \ α	.05	.1	.2	.5	1.	2.	5.	10.
$n = 10$								
U_1	1.9000	.9000	.4000	.1000	.0000	-.0500	-.0800	-.0900
U_2	1.6364	.7273	.2727	.0000	-.0909	-.1364	-.1636	-.1727
U_3	1.4210	.6021	.1918	-.0564	-.1409	-.1845	-.2118	-.2212
U_4	-.0248	-.0467	-.0832	-.1552	-.2143	-.2598	-.2921	-.3032
U_{11}	.0832	.0587	.0178	-.0627	-.1286	-.1786	-.2126	-.2227
$n = 20$								
U_1	.9500	.4500	.2000	.0500	.0000	-.0250	-.0400	-.0450
U_2	.8571	.3810	.1429	.0000	-.0476	-.0714	-.0857	-.0905
U_3	.7890	.3364	.1098	-.0266	-.0726	-.0960	-.1103	-.1152
U_4	-.0173	-.0323	-.0568	-.1025	-.1367	-.1605	-.1762	-.1815
U_{11}	.0343	.0185	-.0073	-.0554	-.0915	-.1165	-.1327	-.1379

Estimator / α	U_1	U_2	U_3	U_4	U_7	U_9	U_{10}	U_{11}	U_{12}	U_{13}	$\sigma^{(2)}$	$\sigma^{(3)}$
n = 10												
.05	7.7000	6.0579	4.7678	.1078	.1756	.1511	.1821	.1394	.1283	.1195	.2045	.1875
.10	1.900	1.4298	1.1023	.1056	.1599	.1443	.1658	.1312	.1246	.1181	.1833	.1694
.20	.5000	.3554	.2738	.1035	.1374	.1336	.1420	.1194	.1192	.1164	.1534	.1438
.30	.2556	.1809	.1464	.1034	.1230	.1230	.1265	.1120	.1158	.1157	.1347	.1277
.40	.1756	.1281	.1107	.1044	.1138	.1138	.1165	.1074	.1137	.1158	.1229	.1174
.50	.1400	.1074	.0982	.1060	.1079	.1074	.1010	.1047	.1127	.1163	.1156	.1109
.60	.1222	.0983	.0937	.1079	.1042	.0983	.1058	.1031	.1124	.1172	.1110	.1069
.70	.1122	.0939	.0923	.1100	.1020	.0939	.1031	.1024	.1126	.1184	.1083	.1045
.80	.1063	.0919	.0923	.1121	.1008	.0919	.1015	.1022	.1131	.1197	.1068	.1031
.90	.1025	.0911	.0929	.1142	.1002	.0911	.1007	.1024	.1139	.1210	.1061	.1025
1.00	.1000	.0909	.0937	.1162	.1000	.0909	.1002	.1028	.1148	.1225	.1059	.1023
2.00	.0950	.0950	.1021	.1309	.1046	.0950	.1037	.1100	.1251	.1355	.1111	.1071
5.00	.0968	.1015	.1111	.1461	.1128	.1015	.1113	.1207	.1389	.1523	.1189	.1153
10.00	.0982	.1043	.1148	.1523	.1162	.1043	.1148	.1253	.1444	.1586	.1207	.1183

contd...

Estimator / α	U_1	U_2	U_3	U_4	U_7	U_9	U_{10}	U_{11}	U_{12}	U_{13}	$\sigma^{(2)}$	$\sigma^{(3)}$
$n = 20$												
.05	1.9500	1.6848	1.4825	.0820	.0776	.0672	.0785	.0583	.0594	.0539	.1004	.088
.10	.5000	.4150	.3592	.0515	.0715	.0653	.0723	.0563	.0578	.0543	.0899	.0798
.20	.1500	.1202	.1046	.0515	.0630	.0612	.0636	.0536	.0555	.0553	.0757	.0686
.30	.0889	.0723	.0652	.0521	.0578	.0578	.0582	.0521	.0542	.0563	.0673	.0619
.40	.0688	.0578	.0540	.0530	.0545	.0545	.0548	.0514	.0535	.0574	.0623	.0579
.50	.0600	.0522	.0500	.0540	.0525	.0522	.0528	.0512	.0532	.0586	.0593	.0554
.60	.0556	.0496	.0485	.0550	.0513	.0496	.0518	.0513	.0532	.0597	.0576	.0540
.70	.0531	.0485	.0480	.0560	.0506	.0485	.0507	.0515	.0533	.0607	.0566	.0532
.80	.0516	.0480	.0479	.0569	.0502	.0479	.0503	.0518	.0536	.0617	.0561	.0527
.90	.0506	.0477	.0481	.0578	.0500	.0477	.0501	.0522	.0539	.0626	.0558	.0525
1.00	.0500	.0476	.0483	.0586	.0500	.0476	.0500	.0525	.0543	.0635	.0558	.0525
2.00	.0488	.0488	.0506	.0639	.0512	.0488	.0511	.0558	.0575	.0693	.0572	.0538
5.00	.0492	.0505	.0530	.0685	.0532	.0505	.0530	.0592	.0609	.0750	.0591	.0558
10.00	.0496	.0513	.0541	.0703	.0541	.0513	.0539	.0606	.0623	.0770	.0595	.0565

0–.2	.25	.30	.35	.40	.45	.5–.7	.75	.8–1.85	1.9	1.95	2–10
U_4	U_4	U_4	U_3	U_3	U_3	U_3	U_2	U_2	U_1	U_1	U_1
U_4	U_4	U_4	U_4	U_3	U_3	U_3	U_3	U_2	U_2	U_1	U_1
U_4	U_4	U_4	U_4	U_4	U_3	U_3	U_3	U_2	U_2	U_2	U_1
U_4	U_4	U_4	U_4	U_4	U_3	U_3	U_3	U_2	U_2	U_2	U_1
U_4	U_4	U_4	U_4	U_4	U_3	U_3	U_3	U_2	U_2	U_2	U_1
U_4	U_4	U_4	U_4	U_4	U_3	U_3	U_3	U_2	U_2	U_2	U_1
U_4	U_4	U_4	U_4	U_{11}	U_{11}	U_3	U_3	U_2	U_2	U_2	U_1
U_4	U_4	U_{11}	U_{11}	U_{11}	U_{11}	U_3	U_3	U_2	U_2	U_2	U_1
U_4	U_{11}	U_{11}	U_{11}	U_{11}	U_{11}	U_3	U_3	U_2	U_2	U_2	U_1

TABLE 4.6.4: Range of α values giving the optimum value m* of m in U_9

n \ m*	n-4	n-3	n-2	n-1	n
3	-	-	.01-.09	.1-.50	.51-
4	-	-	.01-.15	.16-.50	.51-
5	-	-	.01-.18	.19-.50	.51-
6	-	-	.01-.20	.21-.49	.50-
7	-	-	.01-.22	.23-.49	.50-
8	-	-	.01-.23	.24-.49	.50-
9	-	-	.01-.24	.25-.49	.50-
10	-	.01-.02	.03-.24	.25-.49	.50-
11	-	.01-.04	.05-.25	.26-.49	.50-
12	-	.01-.06	.07-.25	.26-.49	.50-
13	-	.01-.07	.08-.26	.27-.49	.50-
14	-	.01-.08	.09-.26	.27-.48	.49-
15	-	.01-.09	.10-.27	.28-.48	.49-
16	-	.01-.10	.11-.27	.28-.48	.49-
17	-	.01-.10	.11-.27	.28-.48	.49-
18	-	.01-.11	.12-.27	.28-.48	.49-
19	-	.01-.12	.13-.27	.28-.48	.49-
20	-	.01-.12	.13-.28	.29-.48	.49-
30	.01-.04	.05-.15	.16-.28	.29-.47	.48-
40	.01-.07	.08-.17	.18-.29	.3-.46	.47-
50	.01-.09	.10-.18	.19-.29	.3-.46	.47-

TABLE 4.7.1: Simulated values of biases of various estimators based on 1000 iterations for n = 10 and 500 iterations for n = 20 when there is one outlier

Est. \ α	.05	.1	.2	.5	1.	2.	5.	10.
n = 10								
U_1	1.9894	.8959	.3934	.0998	.0190	-.0393	-.0823	-.0928
U_2	1.7176	.7236	.2668	-.0003	-.0737	-.1266	-.1657	-.1753
U_3	1.4946	.5987	.1865	-.0568	-.1241	-.1751	-.2138	-.2236
U_4	-.0123	-.0473	-.0860	-.1596	-.1956	-.2486	-.2919	-.3063
U_9	.0991	.0718	.0354	-.0529	-.1012	-.1550	-.1606	-.2134
U_{11}	.0977	.0586	.0142	-.0678	-.1084	-.1663	-.2121	-.2261
mle	.1461	.1444	.1377	.0688	.0135	-.0445	-.0774	-.0959
mmle	1.1551	.6366	.3244	.0973	.0251	-.0360	-.0836	-.0966
T(1)	.0740	.0774	.0517	-.0260	-.0771	-.1304	-.1607	-.1775
T(2)	.0257	.0102	-.0072	-.0729	-.1136	-.1657	-.2106	-.2291
T(3)	.0286	.0157	.0080	-.0548	-.0925	-.1459	-.1929	-.2128
n = 20								
U_1	.9230	.4572	.1803	.0454	.0008	-.0254	-.0391	-.0577
U_2	.8314	.3878	.1241	-.0045	-.0468	-.0718	-.0849	-.1025
U_3	.7646	.3427	.0922	-.0309	-.0719	-.0964	-.1095	-.1269
U_4	-.0126	-.0365	-.0612	-.1093	-.1387	-.1631	-.1765	-.1907
U_9	.0664	.0494	.0257	-.0305	-.0611	-.0869	-.1014	-.1151
U_{11}	.0391	.0145	-.0121	-.0628	-.0939	-.1192	-.1330	-.1477
mle	.0888	.0716	.0585	.0297	-.0035	-.0314	-.0443	-.0606
mmle	.6392	.3555	.1602	.0445	.0016	-.0251	-.0395	-.0581
T(1)	.0252	.0311	.0145	-.0180	-.0500	-.0775	-.0894	-.1049
T(2)	.0167	.0088	.0003	-.0422	-.0672	-.0953	-.1094	-.1210
T(3)	.0168	.0112	.0094	-.0314	-.0560	-.0869	-.1014	-.1111

TABLE 4.7.2: Simulated values of mse of various estimators based on 1000 iterations for n = 10, and 500 iterations for n = 20

Est. \ α	.05	.1	.2	.5	1.	2.	5.	10.
n = 10								
U_1	8.2555	1.9123	.4811	.1452	.1062	.0974	.0944	.1053
U_2	6.5022	1.4406	.3408	.1118	.0929	.0952	.0999	.1107
U_3	5.1219	1.1139	.2618	.1024	.0938	.1014	.1097	.1208
U_4	.1085	.1089	.0976	.1136	.1137	.1280	.1428	.1597
U_9	.1723	.1519	.1206	.1115	.0988	.1041	.1106	.1258
U_{11}	.1436	.1354	.1114	.1129	.1042	.1091	.1167	.1338
mle	.2643	.2511	.2009	.1651	.1203	.1101	.1127	.1231
mmle	2.3027	.8916	.3252	.1434	.1091	.0985	.0942	.1060
T(1)	.1997	.1916	.1481	.1307	.1045	.1056	.1130	.1241
T(2)	.1324	.1289	.1157	.1178	.1051	.1111	.1188	.1354
T(3)	.1331	.1306	.1235	.1236	.0988	.1129	.1182	.1373
n = 20								
U_1	1.7228	.5749	.1468	.0577	.0475	.0500	.0466	.0527
U_2	1.4812	.4823	.1190	.0505	.0453	.0499	.0481	.0553
U_3	1.3003	.4196	.1048	.0487	.0459	.0517	.0506	.0584
U_4	.0512	.0467	.0547	.0540	.0556	.0663	.0665	.0742
U_9	.0722	.0610	.0631	.0488	.0472	.0529	.0512	.0581
U_{11}	.0580	.0506	.0566	.0505	.0489	.0582	.0569	.0639
mle	.1028	.0761	.0781	.0576	.0515	.0563	.0524	.0559
mmle	.7217	.3091	.1205	.0570	.0473	.0502	.0464	.0525
T(1)	.0623	.0584	.0671	.0517	.0492	.0560	.0535	.0582
T(2)	.0596	.0535	.0590	.0501	.0487	.0555	.0541	.0605
T(3)	.0591	.0529	.0614	.0524	.0490	.0565	.0544	.0602

CHAPTER 5

ESTIMATION OF SCALE PARAMETER OF AN EXPONENTIAL DISTRIBUTION IN TWO OUTLIER EXCHANGEABLE MODEL

5.1 Introduction

In this chapter, we consider the estimation of parameters of the exponential distribution when sample contains two outliers under the exchangeable model. Let $X_1,\ldots,X_n$ be a sample of size n in which (n-2) variables follow an exponential distribution with cdf F(x) and pdf given at equation (1.1.1). And the remaining two come from another exponential distribution with cdf G(x) and pdf

$$f(x;\frac{\sigma}{\alpha}) \equiv g(x) = \begin{cases} \frac{\alpha}{\sigma}\exp(-\alpha x/\sigma) & \text{for } x>0,\ \sigma>0,\ \alpha>0 \\ 0 & \text{otherwise.} \end{cases}$$

The joint density of $X_1,\ldots,X_n$ for the exchangeable model is given by

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \frac{\alpha^2}{\sigma^n}\frac{1}{n(n-1)}\, e^{-\frac{n\bar{x}}{\sigma}} \sum_{i\neq j=1}^{n} e^{[(\frac{1-\alpha}{\sigma})(x_i+x_j)]}, \tag{5.1.1}$$

where $\bar{x} = \sum_{i=1}^{n} x_i/n$. Let $X_{(1)},\ldots,X_{(n)}$ be the corresponding order statistics. Then the joint density of $X_{(1)},\ldots,X_{(n)}$ is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)},\ldots,x_{(n)}) = \frac{\alpha^2}{\sigma^n}(n-2)!\, e^{-\frac{n\bar{x}}{\sigma}} \sum_{i\neq j} e^{[\frac{(1-\alpha)}{\sigma}(x_{(i)}+x_{(j)})]}. \tag{5.1.2}$$

We transform $X_{(i)}$ to $Y_{(i)}$ such that

$$Y_{(i)} = \frac{X_{(i)}}{\sigma} \qquad \forall\ i = 1,\ldots,n.$$

This essentially amounts to taking $\sigma = 1$. We consider $\sigma = 1$ throughout this chapter except in Section 5.5 where mle is discussed. The joint density of $Y_{(1)},\ldots,Y_{(n)}$ is given by

$$h_{Y_{(1)},\ldots,Y_{(n)}}(y_{(1)},\ldots,y_{(n)}) = (n-2)!\alpha^2 \bar{e}^{n\bar{y}} \sum_{i\neq j} e^{(1-\alpha)(y_{(i)}+y_{(j)})} \tag{5.1.3}$$

Appropriate references for this situation are given in Section 1.6.

In Sections 5.2 and 5.3, we give the distribution theory for the model described above. In Section 5.4, the correlation coefficient between the smallest and the largest order statistics is obtained and it is shown that it attains local maximum at $\alpha = 1$. This generalizes a similar result due to Gross et al. (1986) for the case of a single outlier. The ml equations and estimators for σ and α are given in Section 5.5. Finally, in last section, we compare the different estimators, and study the robustness of estimators considered in Chapter 4.

We need the following standard results in subsequent sections, for example see Gross et al. (1986). For a positive integer n and $\alpha > 0$,

$$\int_0^\infty x^k \bar{e}^{x\alpha} (1-\bar{e}^{x})^{n-1}\, dx = \int_0^1 (-\log y)^k (y)^{\alpha-1} (1-y)^{n-1}\, dy$$

and

$$\int_0^1 (\log y)^k\, y^{\alpha-1} (1-y)^{n-1}\, dy$$

$$
= \begin{cases}
B(\alpha,n) & \text{for } k = 0 \\[2ex]
-B(\alpha,n) \sum\limits_{j=o}^{n-1} \dfrac{1}{j+\alpha} & \text{for } k = 1 \\[2ex]
B(\alpha,n)\left[\sum\limits_{j=o}^{n-1} \dfrac{1}{(j+\alpha)^2} + \left(\sum\limits_{j=o}^{n-1} \dfrac{1}{j+\alpha} \right)^2 \right] & \text{for } k = 2 \\[2ex]
-B(\alpha,n)\left[\sum\limits_{j=o}^{n-1} \dfrac{2}{(j+\alpha)^3} + 3 \sum\limits_{j=o}^{n-1} \dfrac{1}{j+\alpha} \sum\limits_{j=o}^{n-1} \dfrac{1}{(j+\alpha)^2} + \right. & \\
\qquad \left. \left(\sum\limits_{j=o}^{n-1} \dfrac{1}{j+\alpha} \right)^3 \right] & \text{for } k = 3.
\end{cases}
\tag{5.1.4}
$$

5.2 Distribution theory

The marginal pdf of $Y_{(i)}$ and $(Y_{(i)}, Y_{(j)})$ can be obtained by direct integration of other variables from equation (5.1.3). However, the cdf of $Y_{(i)}$ may also be obtained by direct argument as follows. For $i = 1,2,\ldots,n-2$,

$H_{i:n}(y) = \Pr[Y_{(i)} \leq y]$

$= \Pr[\text{at least } i \text{ of } Y_1,\ldots,Y_n \leq y]$

$= \Pr[\text{at least } i \text{ of } Y_i\text{'s with pdf } f(x) \leq y] +$

$\Pr[\text{exactly } (i-1) \text{ of } Y_i\text{'s with pdf } f(x) \leq y$ and one with pdf $g(x) \leq y] + \Pr[\text{exactly } (i-2) \text{ of } Y_i\text{'s}$ with pdf $f(x) \leq y$ and two with pdf $g(x) \leq y]$

$$
= \sum_{r=i}^{n} \binom{n-2}{r-2} F^{r-2}(y)(1-F(y))^{n-r} G^2(y) +
$$

$$
\sum_{r=i}^{n-1} \binom{n-2}{r-1} F^{r-1}(y)(1-F(y))^{n-r-1} \binom{2}{1} G(y)(1-G(y)) +
$$

$$
\sum^{n-2} \binom{n-2}{\;} F^{r}(y)(1-F(y))^{n-r-2}(1-G(y))^2
$$

$$H_{n-1:n}(y) = \sum_{r=n-1}^{n} \binom{n-2}{r-2} F^{r-2}(y)(1-F(y))^{n-r} G^2(y) +$$
$$2F^{n-2}(y)\ G(y)(1-G(y)),$$

$$H_{n:n}(y) = F^{n-2}(y)\ G^2(y). \tag{5.2.1}$$

Substituting for $F(y)$, $G(y)$ and differentiating w.r.t. y, we get the pdf of $Y_{(i)}$ as

$$\begin{aligned} h_{i:n}(y) = {} & \frac{(n-2)!}{(i-3)!(n-i)!} (1-\bar{e}^{y})^{i-3} (1-\bar{e}^{\alpha y})^2 (\bar{e}^{y})^{n-i+1} + \\ & \frac{2(n-2)!}{(i-2)!(n-i)!} (1-\bar{e}^{y})^{i-2} (1-\bar{e}^{\alpha y}) (\bar{e}^{y})^{n-i} \alpha \bar{e}^{\alpha y} + \\ & \frac{2(n-2)!}{(i-2)!(n-i-1)!} (1-\bar{e}^{y})^{i-2} (1-\bar{e}^{\alpha y}) (\bar{e}^{y})^{n-i} \bar{e}^{\alpha y} + \\ & \frac{2(n-2)!}{(i-1)!(n-i-1)!} (1-\bar{e}^{y})^{i-1} (\bar{e}^{y})^{n-i-1} \alpha \bar{e}^{2\alpha y} + \\ & \frac{(n-2)!}{(i-1)!(n-i-2)!} (1-\bar{e}^{y})^{i-1} (\bar{e}^{y})^{n-i-1} \bar{e}^{2\alpha y}, \end{aligned} \tag{5.2.2}$$

where for $i = 1$, first, second and third, for $i = n-1$, last and for $i = n$, third, fourth and fifth term drop out. Thus $h_{1:n}(y)$ is

$$h_{1:n}(y) = (n+2\alpha-2)(\bar{e}^{y})^{n+2\alpha-2} \qquad 0 < y < \infty,$$

which shows that $Y_{(1)}$ has an exponential distribution with mean and variance given by

$$E(Y_{(1)}) = \frac{1}{n+2\alpha-2}$$

and

$$V(Y_{(1)}) = \frac{1}{(n+2\alpha-2)^2}\ . \tag{5.2.3}$$

Now we evaluate other moments of $Y_{(i)}$ for the pdf given in equation (5.2.2). Using equations (2.1.5) and (5.2.1), we

$$E(Y_{(n)}) = \int_0^\infty (1-(1-\bar{e}^{y})^{n-2} (1-\bar{e}^{\alpha y})^2)\, dy$$

$$= \int_0^\infty [\{1-(1-\bar{e}^{y})^{n-2}\} - (1-\bar{e}^{y})^{n-2}\, \bar{e}^{2\alpha y} + 2(1-\bar{e}^{y})^{n-2}\, \bar{e}^{\alpha y}]\, dy.$$

Using equation (5.1.4), it gives

$$E(Y_{(n)}) = \sum_{i=1}^{n-2} \frac{1}{i} - B(2\alpha, n-1) + 2B(\alpha, n-1). \qquad (5.2.4)$$

For other values of r, on using equations (2.1.5) and (5.2.1), we get

$$E(Y_{(r)}) = \int_0^\infty [1 - \sum_{i=r}^{n} \binom{n-2}{i-2} (1-\bar{e}^{y})^{i-2} (1-\bar{e}^{\alpha y})^2 (\bar{e}^{y})^{n-i} -$$

$$2 \sum_{i=r}^{n-1} \binom{n-2}{i-1} (1-\bar{e}^{y})^{i-1}\, \bar{e}^{\alpha y} (1-\bar{e}^{\alpha y}) (\bar{e}^{y})^{n-i-1} -$$

$$\sum_{i=r}^{n-2} \binom{n-2}{i} (1-\bar{e}^{y})^{i}\, \bar{e}^{2\alpha y} (\bar{e}^{y})^{n-i-2}]\, dy .$$

This can be written as

$$E(Y_{(r)}) = \int_0^\infty [1 - \sum_{i=r-2}^{n-2} \binom{n-2}{i} (1-\bar{e}^{y})^{i} (\bar{e}^{y})^{n-i-2} -$$

$$\sum_{i=r-2}^{n-2} \binom{n-2}{i} (1-\bar{e}^{y})^{i} (\bar{e}^{y})^{n-i-2+2\alpha} +$$

$$2 \sum_{i=r-2}^{n-2} \binom{n-2}{i} (1-\bar{e}^{y})^{i} (\bar{e}^{y})^{n-i-2+\alpha} -$$

$$2 \sum_{i=r-1}^{n-2} \binom{n-2}{i} (1-\bar{e}^{y})^{i} (\bar{e}^{y})^{n-i-2+\alpha} +$$

$$2 \sum_{i=r-1}^{n-2} \binom{n-2}{i} (1-\bar{e}^{y})^{i} (\bar{e}^{y})^{n-i-2+2\alpha} -$$

$$\sum_{i=r}^{n-2} \binom{n-2}{i} (1-\bar{e}^{y})^{i} (\bar{e}^{y})^{n-i-2+2\alpha}]dy$$

$$= \sum_{i=1}^{r-2} \frac{1}{n-1-i} - \sum_{i=r-2}^{n-2} \binom{n-2}{i} B(i+1,n-i-2+2\alpha) +$$

$$2 \sum_{i=r-2}^{n-2} \binom{n-2}{i} B(i+1,n-i-2+\alpha) - 2 \sum_{i=r-1}^{n-2} \binom{n-2}{i} B(i+1,n-i-2+\alpha)$$

$$- \sum_{i=r}^{n-2} \binom{n-2}{i} B(i+1,n-i-2+2\alpha) + 2 \sum_{i=r-1}^{n-2} \binom{n-2}{i} B(i+1,n-i-2+2\alpha),$$

on using equation (5.1.4). After some simplification, it reduces to

$$E(Y_{(r)}) = \sum_{i=1}^{r-2} \frac{1}{n-1-i} + \frac{2(n-2)!\,\Gamma(n-r+\alpha)}{\Gamma(n+\alpha-1)\,\Gamma(n-r+1)} - \frac{\Gamma(n-1)\,\Gamma(n-r+2\alpha)}{\Gamma(n-r+1)\,\Gamma(n+2\alpha-1)} +$$

$$\frac{\Gamma(n-1)\,\Gamma(n-r+2\alpha-1)}{\Gamma(n-r)\,\Gamma(n+2\alpha-1)} . \qquad (5.2.5)$$

Higher order moments can be obtained in a similar manner. Thus for finding $Var(Y_{(n)})$, note that equations (2.1.5) and (5.2.1) give

$$E(Y^2_{(n)}) = \int_0^\infty 2y[1 - (1-\bar{e}^{y})^{n-2} (1-\bar{e}^{\alpha y})^2]dy$$

$$= \int_0^\infty 2y[1-(1-\bar{e}^{y})^{n-2} - (1-\bar{e}^{y})^{n-2} \bar{e}^{2\alpha y} + 2(1-\bar{e}^{y})^{n-2} \bar{e}^{\alpha y}]dy$$

$$= \sum_{i=1}^{n-2} \frac{1}{i^2} + \left(\sum_{i=1}^{n-2} \frac{1}{i}\right)^2 - 2B(2\alpha,n-1) \sum_{i=1}^{n-1} \frac{1}{2\alpha+i-1} +$$

$$4B(\alpha,n-1) \sum_{i=1}^{n-1} \frac{1}{\alpha+i-1} , \qquad (5.2.6)$$

on using equation (5.1.4). Using equations (5.2.4) and (5.2.6), we obtain $Var(Y_{(n)})$.

For other values of r, again we start with equations (2.1.5) and (5.2.1), which give

$$E(Y^2_{(r)}) = \int_0^\infty 2y[1 - \sum_{i=r-2}^{n-2} \binom{n-2}{i}(1-e^{-y})^i (1-e^{-\alpha y})^2 (e^{-y})^{n-i-2} -$$

$$2 \sum_{i=r-1}^{n-2} \binom{n-2}{i}(1-e^{-y})^i (1-e^{-\alpha y}) (e^{-y})^{n-i-2} e^{-\alpha y} -$$

$$\sum_{i=r}^{n-2} \binom{n-2}{i}(1-e^{-y})^i (e^{-y})^{n-i-2} e^{-2\alpha y}]dy .$$

On using equation (5.1.4), we get

$$E(Y^2_{(r)}) = \sum_{i=1}^{r-2} \frac{1}{(n-2-i+1)^2} + \left(\sum_{i=1}^{r-2} \frac{1}{n-i-1}\right)^2 - 2 \sum_{i=r-2}^{n-2} \binom{n-2}{i}$$

$$\cdot \{B(i+1,n-i-2+2\alpha) \sum_{j=1}^{i+1} \frac{1}{n-i+2\alpha-2+j-1} -$$

$$2B(i+1,n-i-2+\alpha) \sum_{j=1}^{i+1} \frac{1}{n-i+\alpha+j-3}\} -$$

$$4 \sum_{i=r-1}^{n-2} \binom{n-2}{i} \{B(i+1,n-2-i+\alpha) \sum_{j=1}^{i+1} \frac{1}{n+\alpha-i+j-3} -$$

$$B(i+1,n-2-i+2\alpha) \sum_{j=1}^{i+1} \frac{1}{n-i+2\alpha+j-3}\} -$$

$$2 \sum_{i=r}^{n-2} \binom{n-2}{i} B(i+1,n-i-2+2\alpha) \sum_{j=1}^{i+1} \frac{1}{n+2\alpha-3-i+j} .$$

After simplification, it gives

$$E(Y^2_{(r)}) = \sum_{i=1}^{r-2} \frac{1}{(n-i-1)^2} + \left(\sum_{i=1}^{r-2} \frac{1}{n-i-1}\right)^2 + \frac{4\Gamma(n-r+\alpha)\Gamma(n-1)}{\Gamma(n-r+1)\Gamma(n+\alpha-1)} \sum_{i=1}^{r-1}$$

$$\cdot \frac{1}{n-r+\alpha+i-1} + \frac{2\Gamma(n-1)\Gamma(n+2\alpha-r-1)}{\Gamma(n-r)\quad (n+2\alpha-r-1)} \frac{1}{\Gamma(n+2\alpha-1)} -$$

$$(2\alpha-1) \sum_{i=1}^{r-1} \frac{1}{n-r+2\alpha+i-1} \frac{2\Gamma(n-1)\Gamma(n+2\alpha-r-1)}{\Gamma(n+2\alpha-1)\Gamma(n-r+1)} . \qquad (5.2.7)$$

This direct argument provides means and variances of order statistics. But if we try to obtain $E(Y_{(r)}Y_{(s)})$ for $r < s$, expressions become very complicated. We, therefore, use alternative technique, which is similar to the one considered by Joshi (1972), for determining product moments. This is discussed in the next section.

5.3 Moments of order statistics using alternative method

Now we use the transformation given in equation (4.3.1) for this case. Using equation (5.1.3), the joint pdf of $Z_1,\ldots,Z_n$ is given by

$$h_{Z_1,\ldots,Z_n}(z_1,\ldots,z_n)$$

$$= \frac{\alpha^2}{n(n-1)} e^{-\sum_{i=1}^{n} z_i} \sum_{i\neq j} e^{(1-\alpha)\left(\sum_{r=1}^{i} \frac{z_r}{n-r+1} + \sum_{r=1}^{j} \frac{z_r}{n-r+1}\right)}$$

$$= \frac{\alpha^2}{n(n-1)} e^{-\sum_1^n z_i}\, 2 \sum_{i<j} e^{(1-\alpha)\left(2\sum_{r=1}^{i} \frac{z_r}{n-r+1} + \sum_{r=i+1}^{j} \frac{z_r}{n-r+1}\right)}$$

$$= \frac{2\alpha^2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} e^{-\sum_{r=1}^{i} z_r\left(1-\frac{2(1-\alpha)}{n-r+1}\right) - \sum_{r=i+1}^{j} z_r\left(1-\frac{1-\alpha}{n-r+1}\right) - \sum_{r=j+1}^{n} z_r}$$

$$= \frac{2\alpha^2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} e^{-\sum_{r=1}^{i} a_r z_r - \sum_{r=i+1}^{j} b_r z_r - \sum_{r=j+1}^{n} z_r},$$

$$0 \le z_1,\ldots,z_n \le \infty, \qquad (5.3.1)$$

where $a_r = \dfrac{n-r+2\alpha-1}{n-r+1}$

and $b_r = \dfrac{n-r+\alpha}{n-r+1}$, which is same as b_i given in Section 4.3.

This can be expanded as

$h_{Z_1,..,Z_n}(z_1,..,z_n)$

$$= \frac{2\alpha^2}{n(n-1)} [\{e^{-a_1z_1-b_2z_2-\sum_{r=3}^{n} z_r} + e^{-a_1z_1-\sum_{r=2}^{3} b_r z_r - \sum_{r=4}^{n} z_r} + \ldots +$$

$$e^{-a_1z_1-\sum_{r=2}^{n} b_r z_r}\} + \{e^{-a_1z_1-a_2z_2-b_3z_3-\sum_{r=4}^{n} z_r} +$$

$$e^{-a_1z_1-a_2z_2-\sum_{r=3}^{4} b_r z_r - \sum_{r=5}^{n} z_r} + \ldots\} + \ldots + \{e^{-\sum_{r=1}^{n-1} a_r z_r - b_n z_n}\}]$$

$$= a_1 e^{-a_1z_1} h_{Z_2,..,Z_n}(z_2,..,z_n) , \qquad (5.3.2)$$

where $a_1e^{-a_1z_1}$ is the pdf of Z_1 and $h_{Z_2,..,Z_n}(z_2,..,z_n)$ is the pdf of $(Z_2,..,Z_n)$. This immediately shows that Z_1 and $(Z_2,..,Z_n)$ are independently distributed. This result has also been established by Gross et al. (1986), where they have considered a more general situation.

Integrating out $Z_1,Z_2,...,Z_{i-1},Z_{i+1},...,Z_n$, we get the marginal density of Z_i as

$$h_{Z_i}(z_i) = h_1(z_i) + h_2(z_i), \qquad 0 < z_i < \infty , \qquad (5.3.3)$$

where

$$h_1(z_i) = \frac{2\alpha^2}{n(n-1)}[\{\frac{e^{-z_i}}{a_1b_2} + \frac{e^{-z_i}}{a_1b_2b_3} + \ldots + \frac{e^{-z_i}}{a_1b_2\ldots b_{i-1}} + \frac{e^{-b_iz_i}}{a_1b_2\ldots b_{i-1}} +$$

$$\frac{e^{-b_iz_i}}{a_1b_2\ldots b_{i-1}b_{i+1}} + \ldots + \frac{e^{-b_iz_i}}{a_1b_2\ldots b_{i-1}b_{i+1}\ldots b_n}\} + \{\frac{e^{-z_i}}{a_1a_2b_3} +$$

$$\frac{e^{-z_i}}{a_1a_2b_3b_4} + \ldots + \frac{e^{-b_iz_i}}{a_1a_2b_3\ldots b_{i-1}} + \ldots + \frac{e^{-b_iz_i}}{a_1a_2b_3\ldots b_{i-1}b_{i+1}\ldots b_n}\}$$

$$+ \ldots + \{\frac{e^{-b_iz_i}}{a_1\ldots a_{i-1}} + \frac{e^{-b_iz_i}}{a_1\ldots a_{i-1}b_{i+1}} + \ldots\}]$$

and

$$h_2(z_i) = \frac{2\alpha^2}{n(n-1)} \Big[\Big\{ \frac{e^{-a_i z_i}}{a_1 \cdots a_{i-1} b_{i+1}} + \frac{e^{-a_i z_i}}{a_1 \cdots a_{i-1} b_{i+1} b_{i+2}} + \cdots + \frac{e^{-a_i z_i}}{a_1 \cdots a_{i-1} b_{i+1} \cdots b_n} \Big\} + \Big\{ \frac{e^{-a_i z_i}}{a_1 \cdots a_{i-1} a_{i+1} b_{i+2}} + \cdots + \frac{e^{-a_i z_i}}{a_1 \cdots a_{i-1} a_{i+1} b_{i+2} \cdots b_n} \Big\} + \cdots + \Big\{ \frac{e^{-a_i z_i}}{a_1 \cdots a_{n-1} b_n} \Big\} \Big].$$

Here $h_1(z_i)$ drops out for $i = 1$ and $h_2(z_i)$ drops out for $i = n$. This convention is followed in evaluating the expected values of Z_i and Z_i^2. Now

$$E(Z_i) = \int_0^\infty z_i \, h_{Z_i}(z_i) \, dz_i$$

$$= \frac{2\alpha^2}{n(n-1)} \Big[\Big\{ \frac{1}{a_1 b_2} + \frac{1}{a_1 b_2 b_3} + \cdots + \frac{1}{a_1 b_2 \cdots b_{i-1} b_i^2} + \frac{1}{a_1 b_2 \cdots b_i^2 b_{i+1}} + \cdots + \frac{1}{a_1 b_2 \cdots b_i^2 \cdots b_n} \Big\} + \Big\{ \frac{1}{a_1 a_2 b_3} + \frac{1}{a_1 a_2 b_3 b_4} + \cdots + \frac{1}{a_1 a_2 b_3 \cdots b_i^2 \cdots b_n} \Big\} + \cdots + \Big\{ \frac{1}{a_1 a_2 \cdots a_{i-1} b_i^2} + \cdots + \frac{1}{a_1 a_2 \cdots a_{i-1} b_i^2 b_{i+1} \cdots b_n} \Big\} + \Big\{ \frac{1}{a_1 \cdots a_i^2 b_{i+1}} + \frac{1}{a_1 \cdots a_i^2 b_{i+1} b_{i+2}} + \cdots + \frac{1}{a_1 \cdots a_i^2 b_{i+1} \cdots b_n} \Big\} + \Big\{ \frac{1}{a_1 \cdots a_i^2 a_{i+1} b_{i+2}} + \frac{1}{a_1 \cdots a_i^2 a_{i+1} b_{i+2} b_{i+3}} + \cdots \Big\} + \cdots + \Big\{ \frac{1}{a_1 a_2 \cdots a_i^2 \cdots a_{n-1} b_n} \Big\} \Big],$$

after simple integrations. This after some rearrangement of terms, becomes

$$E(Z_i) = \frac{2\alpha^2}{n(n-1)} \Big[\frac{b_1}{a_1} \Big\{\frac{1}{b_1 b_2} + \dots + \frac{1}{b_1 b_2 \dots b_{i-1}} + \frac{1}{b_i}\Big(\frac{1}{b_1 \dots b_i} + \dots + \frac{1}{b_1 \dots b_n}\Big)\Big\} + \frac{b_1 b_2}{a_1 a_2} \Big\{\frac{1}{b_1 b_2 b_3} + \dots + \frac{1}{b_1 \dots b_{i-1}} + \frac{1}{b_i}\Big(\frac{1}{b_1 \dots b_i} + \dots + \frac{1}{b_1 \dots b_n}\Big)\Big\} + \dots + \frac{b_1 \dots b_{i-1}}{a_1 \dots a_{i-1}} \Big\{\frac{1}{b_i}\Big(\frac{1}{b_1 \dots b_i} + \dots + \frac{1}{b_1 \dots b_n}\Big)\Big\} + \frac{b_1 \dots b_i}{a_1 \dots a_i^2} \Big\{\frac{1}{b_1 \dots b_{i+1}} + \dots + \frac{1}{b_1 \dots b_n}\Big\} + \dots + \frac{b_1 \dots b_{n-1}}{a_1 \dots a_i^2 \dots a_{n-1}} \Big\{\frac{1}{b_1 \dots b_n}\Big\}\Big]$$

$$= \frac{2\alpha}{(n-1)} \Big[\frac{b_1}{a_1} \Big\{\frac{\alpha}{nb_1 b_2} + \dots + \frac{\alpha}{nb_1 \dots b_{i-1}} + \frac{1}{b_i}\Big(\frac{\alpha}{nb_1 \dots b_i} + \dots + \frac{\alpha}{nb_1 \dots b_n}\Big)\Big\} + \frac{b_1 b_2}{a_1 a_2} \Big\{\frac{\alpha}{nb_1 b_2 b_3} + \dots + \frac{\alpha}{nb_1 \dots b_{i-1}} + \frac{1}{b_i}\Big(\frac{\alpha}{nb_1 \dots b_i} + \dots + \frac{\alpha}{nb_1 \dots b_n}\Big)\Big\} + \dots + \frac{b_1 \dots b_i}{a_1 \dots a_i^2} \Big\{\frac{\alpha}{nb_1 \dots b_{i+1}} + \dots + \frac{\alpha}{nb_1 \dots b_n}\Big\} + \dots + \frac{b_1 \dots b_{n-1}}{a_1 \dots a_{n-1}} \Big\{\frac{\alpha}{nb_1 \dots b_n}\Big\}\Big].$$

Now using equation (4.3.2), we get

$$E(Z_i) = \frac{2\alpha}{(n-1)} \Big[\frac{b_1}{a_1} \Big\{\sum_{r=2}^{i-1} p_r + \frac{1}{b_i} \sum_{r=i}^{n} p_r\Big\} + \frac{b_1 b_2}{a_1 a_2} \Big\{\sum_{r=3}^{i-1} p_r + \frac{1}{b_i} \sum_{r=i}^{n} p_r\Big\} + \dots + \frac{b_1 \dots b_{i-1}}{a_1 \dots a_{i-1}} \Big\{\frac{1}{b_i} \sum_{r=i}^{n} p_r\Big\} + \frac{b_1 \dots b_i}{a_1 \dots a_i^2} \sum_{r=i+1}^{n} p_r + \frac{b_1 \dots b_{i+1}}{a_1 \dots a_i^2 a_{i+1}} \sum_{r=i+2}^{n} p_r + \dots + \frac{b_1 \dots b_{n-1}}{a_1 \dots a_{n-1}} p_n\Big].$$

Using equation (4.3.3), it reduces to

$$E(Z_i) = \frac{2\alpha}{(n-1)} \Big[\frac{b_1}{a_1} \Big\{\frac{n-2+\alpha}{\alpha} p_2 + \frac{1-\alpha}{\alpha} p_i\Big\} + \frac{b_1 b_2}{a_1 a_2} \Big\{\frac{n-3+\alpha}{\alpha} p_3 + \frac{1-\alpha}{\alpha} p_i\Big\} +$$

$$\frac{b_1 b_2 b_3}{a_1 a_2 a_3} \left\{\frac{n-4+\alpha}{\alpha} p_4 + \frac{1-\alpha}{\alpha} p_i\right\} + \dots + \frac{b_1 \dots b_{i-1}}{a_1 \dots a_{i-1}} \left\{\frac{n-i+\alpha}{\alpha} p_i + \frac{1-\alpha}{\alpha} p_i\right\} + \frac{b_1 \dots b_i}{a_1 \dots a_i^2} \frac{(n-i-1+\alpha)}{\alpha} p_{i+1} + \frac{b_1 \dots b_{i+1}}{a_1 \dots a_i^2 a_{i+1}} \frac{(n-i-2+\alpha)}{\alpha} \cdot p_{i+2} + \dots + \frac{b_1 \dots b_{n-1}}{a_1 \dots a_i^2 a_{i+1} \dots a_{n-1}} p_n \Big].$$

Using equation (4.3.2), we get

$$E(Z_i) = \frac{2\alpha}{n(n-1)} \Big[\left\{\frac{n-1}{a_1} + \frac{(n-2)}{a_1 a_2} + \frac{(n-3)}{a_1 a_2 a_3} + \dots + \frac{(n-i+1)}{a_1 \dots a_{i-1}}\right\} + np_i \frac{(1-\alpha)}{\alpha} \left(\frac{b_1}{a_1} + \frac{b_1 b_2}{a_1 a_2} + \dots + \frac{b_1 \dots b_{i-1}}{a_1 \dots a_{i-1}}\right) + \frac{1}{a_1 \dots a_i^2} \left\{\frac{n-i}{1} + \frac{n-i-1}{a_{i+1}} + \dots + \frac{1}{a_{i+1} \dots a_{n-1}}\right\}\Big] \tag{5.3.4}$$

$$= \frac{2\alpha}{n(n-1)} \Big[\frac{n\,\Gamma(n)}{\Gamma(n+2\alpha-1)} \left\{\frac{\Gamma(n+2\alpha-2)}{\Gamma(n-1)} + \frac{\Gamma(n+2\alpha-3)}{\Gamma(n-2)} + \dots + \frac{\Gamma(n+2\alpha-i)}{\Gamma(n-i+1)}\right\} + np_i \frac{(1-\alpha)}{\alpha} \left\{\frac{n-1+\alpha}{n-2+2\alpha} + \frac{(n-1+\alpha)(n-2+\alpha)}{(n-2+2\alpha)(n-3+2\alpha)} + \dots\right\} + \frac{(n-i)}{a_1 \dots a_i^2} \frac{\Gamma(n-i)}{\Gamma(n-i+2\alpha-1)} \left\{\frac{\Gamma(n-i+2\alpha-1)}{\Gamma(n-i)} + \frac{\Gamma(n-i+2\alpha-2)}{\Gamma(n-i-1)} + \dots + \frac{\Gamma(2\alpha)}{\Gamma(1)}\right\}\Big].$$

Using equation (4.3.6), it reduces to

$$E(Z_i) = \frac{2\alpha}{n(n-1)} \Big[\frac{n\,\Gamma(n)}{\Gamma(n+2\alpha-1)} \frac{1}{2\alpha} \left\{\frac{\Gamma(n+2\alpha-1)}{\Gamma(n-1)} - \frac{\Gamma(n+2\alpha-i)}{\Gamma(n-i)}\right\} + \frac{(1-\alpha)}{\alpha} \frac{\Gamma(n+\alpha)}{\Gamma(n-1+2\alpha)} np_i \left\{\frac{\Gamma(n+2\alpha-2)}{\Gamma(n+\alpha-1)} + \dots + \frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i+1)}\right\} + \frac{(n-i)}{a_1 \dots a_i^2} \frac{\Gamma(n-i)}{\Gamma(n-i+2\alpha-1)} \Big[\left\{\frac{\Gamma(n-i+2\alpha)}{\Gamma(n-i)} - \frac{\Gamma(2\alpha+1)}{\Gamma(1)}\right\} \frac{1}{2\alpha} + \frac{\Gamma(2\alpha)}{\Gamma(2)}\Big]\Big]$$

$$= \frac{2\alpha}{n(n-1)} \left[\frac{n(n-1)}{2\alpha} - \frac{n\,\Gamma(n)\,\Gamma(n+2\alpha-i)}{2\alpha\,\Gamma(n+2\alpha-1)\,\Gamma(n-i)} + \frac{(1-\alpha)}{\alpha}\, np_i\, \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)}\right.$$

$$\left. \cdot \frac{1}{\alpha} \left\{\frac{\Gamma(n+2\alpha-1)}{\Gamma(n+\alpha-1)} - \frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i)}\right\} + \frac{(n-i)(n+2\alpha-i-1)}{2\alpha\; a_1 \cdots a_i^2}\right]$$

$$= \frac{2}{n(n-1)} \left[\frac{n(n-1)}{2} - \frac{\Gamma(n+1)\,\Gamma(n+2\alpha-i)}{2\,\Gamma(n+2\alpha-1)\,\Gamma(n-i)} + \frac{(1-\alpha)}{\alpha}\,(n+\alpha-1)np_i -\right.$$

$$\left. \frac{np_i\,\Gamma(n+\alpha)\,\Gamma(n+2\alpha-i)}{\Gamma(n+2\alpha-1)\,\Gamma(n+\alpha-i)} \left(\frac{1-\alpha}{\alpha}\right) + \frac{(n-i)(n+2\alpha-i-1)}{2a_1 \cdots a_i^2}\right].$$

Therefore

$$E(Z_i) = 1 - \frac{\Gamma(n-1)\,\Gamma(n+2\alpha-i)}{\Gamma(n+2\alpha-1)\,\Gamma(n-i)} + \frac{2(1-\alpha)(n+\alpha-1)p_i}{\alpha(n-1)} -$$

$$\frac{2(1-\alpha)\,\Gamma(n+\alpha)\,\Gamma(n+2\alpha-i)}{\alpha(n-1)\,\Gamma(n+2\alpha-1)\,\Gamma(n+\alpha-i)}\, p_i + \frac{(n-i)(n+2\alpha-i-1)}{n(n-1)\; a_1 \cdots a_{i-1} a_i^2}. \tag{5.3.5}$$

We next evaluate $E(Z_i^2)$. Now

$$E(Z_i^2) = \int_0^\infty z_i^2\, h_{Z_i}(z_i)\, dz_i\;,$$

where $h(z_i)$ is given at equation (5.3.3). After simple integrations, we get

$$E(Z_i^2) = \frac{2\alpha^2}{n(n-1)} \left[\left\{\frac{2}{a_1 b_2} + \frac{2}{a_1 b_2 b_3} + \cdots + \frac{2}{a_1 b_2 \cdots b_{i-1}} + \frac{2}{a_1 b_2 \cdots b_{i-1} b_i^3}\right.\right.$$

$$\left. + \frac{2}{a_1 b_2 \cdots b_i^3 b_{i+1}} + \cdots + \frac{2}{a_1 b_2 \cdots b_{i-1} b_i^3 \cdots b_n}\right\} + \left\{\frac{2}{a_1 a_2 b_3} + \right.$$

$$\left. \frac{2}{a_1 a_2 b_3 b_4} + \cdots + \frac{2}{a_1 a_2 b_3 \cdots b_{i-1} b_i^3} + \cdots + \frac{2}{a_1 a_2 b_3 \cdots b_i^3 \cdots b_n}\right\}$$

$$+ \cdots + \frac{2}{a_1 \cdots a_{i-1} b_i^3} + \cdots + \frac{2}{a_1 \cdots a_{i-1} b_i^3 b_{i+1} \cdots b_n}\Big\} +$$

$$\{\frac{2}{a_1a_2\cdots a_i^3b_{i+1}} + \frac{2}{a_1\cdots a_i^3b_{i+1}b_{i+2}} + \cdots + \frac{2}{a_1\cdots a_i^3b_{i+1}\cdots b_n}\}$$

$$+ \{\frac{2}{a_1a_2\cdots a_i^3a_{i+1}b_{i+2}} + \cdots\} + \cdots + \{\frac{2}{a_1\cdots a_i^3a_{i+1}\cdots a_{n-1}b_n}\}].$$

After some simple rearrangement of terms, this becomes

$$E(Z_i^2) = \frac{4\alpha^2}{n(n-1)} [\frac{b_1}{a_1} \{\frac{1}{b_1b_2} + \frac{1}{b_1b_2b_3} + \cdots + \frac{1}{b_1\cdots b_{i-1}} + \frac{1}{b_i^2}(\frac{1}{b_1\cdots b_i}$$

$$+ \cdots + \frac{1}{b_1\cdots b_n})\} + \frac{b_1b_2}{a_1a_2} \{\frac{1}{b_1b_2b_3} + \cdots + \frac{1}{b_1\cdots b_{i-1}} +$$

$$\frac{1}{b_i^2}(\frac{1}{b_1\cdots b_i} + \cdots + \frac{1}{b_1\cdots b_n})\} + \cdots + \frac{b_1\cdots b_{i-1}}{a_1\cdots a_{i-1}} \{\frac{1}{b_i^2}(\frac{1}{b_1\cdots b_i}$$

$$+ \cdots + \frac{1}{b_1\cdots b_n})\} + \frac{b_1\cdots b_i}{a_1\cdots a_i^3} \{\frac{1}{b_1\cdots b_{i+1}} + \frac{1}{b_1\cdots b_{i+1}b_{i+2}}$$

$$+ \cdots\} + \cdots + \frac{b_1\cdots b_{n-1}}{a_1\cdots a_{n-1}} \frac{1}{b_1\cdots b_n}]$$

$$= \frac{4\alpha}{(n-1)}[\frac{b_1}{a_1} \{\frac{\alpha}{nb_1b_2} + \frac{\alpha}{nb_1b_2b_3} + \cdots + \frac{\alpha}{nb_1\cdots b_{i-1}} +$$

$$\frac{1}{b_i^2}(\frac{\alpha}{nb_1\cdots b_i} + \cdots + \frac{\alpha}{nb_1\cdots b_n})\} + \frac{b_1b_2}{a_1a_2} \{\frac{\alpha}{nb_1b_2b_3} + \cdots +$$

$$\frac{\alpha}{nb_1\cdots b_{i-1}} + \frac{1}{b_i^2}(\frac{\alpha}{nb_1\cdots b_i} + \cdots + \frac{\alpha}{nb_1\cdots b_n})\} + \cdots +$$

$$\frac{b_1\cdots b_i}{a_1\cdots a_i^3} \{\frac{\alpha}{nb_1\cdots b_{i+1}} + \frac{\alpha}{nb_1\cdots b_{i+1}b_{i+2}} + \cdots\} + \cdots +$$

$$\frac{b_1\cdots b_{n-1}}{a_1\cdots a_{n-1}} \frac{\alpha}{nb_1\cdots b_n}].$$

On using equation (4.3.2), we have

$$E(z_i^2) = \frac{4\alpha}{(n-1)} \left[\frac{b_1}{a_1} \left\{\sum_{k=2}^{i-1} p_k + \frac{1}{b_i^2} \sum_{k=i}^{n} p_k\right\} + \frac{b_1 b_2}{a_1 a_2} \left\{\sum_{k=3}^{i-1} p_k + \right.\right.$$

$$\left.\frac{1}{b_i^2} \sum_{k=i}^{n} p_k\right\} + \ldots + \frac{b_1 \ldots b_{i-1}}{a_1 \ldots a_{i-1}} \left\{\frac{1}{b_i^2} \sum_{k=i}^{n} p_k\right\} + \frac{b_1 \ldots b_i}{a_1 \ldots a_i^3} \sum_{k=i+1}^{n} p_k$$

$$\left. + \ldots + \frac{b_1 \ldots b_{n-1}}{a_1 \ldots a_{n-1}} p_n\right].$$

Using equation (4.3.3), it gives

$$E(z_i^2) = \frac{4\alpha}{(n-1)} \left[\frac{b_1}{a_1} \left\{\frac{n-2+\alpha}{\alpha} p_2 - \frac{n-i+\alpha}{\alpha} p_i + \frac{1}{b_i^2} \frac{(n-i+\alpha)}{\alpha} p_i\right\} + \right.$$

$$\frac{b_1 b_2}{a_1 a_2} \left\{\frac{n-3+\alpha}{\alpha} p_3 - \frac{n-i+\alpha}{\alpha} p_i + \frac{1}{b_i^2} \frac{(n-i+\alpha)}{\alpha} p_i\right\} + \ldots +$$

$$\frac{b_1 \ldots b_{i-1}}{a_1 \ldots a_{i-1}} \left\{\frac{1}{b_i^2} \frac{(n-i+\alpha)}{\alpha} p_i\right\} + \frac{b_1 \ldots b_i}{a_1 \ldots a_i^3} \frac{(n-i-1+\alpha)}{\alpha} p_{i+1} +$$

$$\left.\frac{b_1 \ldots b_{i+1}}{a_1 \ldots a_i^3 a_{i+1}} \frac{(n-i-2+\alpha)}{\alpha} p_{i+2} + \ldots + \frac{b_1 \ldots b_{n-1}}{a_1 \ldots a_{n-1}} p_n\right]$$

$$= \frac{4\alpha}{(n-1)} \left[\frac{b_1}{a_1} \left\{\frac{n-2+\alpha}{\alpha} p_2 + \frac{(1-\alpha)}{\alpha} \frac{(2n-2i+\alpha+1)}{(n-i+\alpha)} p_i\right\} + \right.$$

$$\frac{b_1 b_2}{a_1 a_2} \left\{\frac{n-3+\alpha}{\alpha} p_3 + \frac{(1-\alpha)}{\alpha} \frac{(2n-2i+\alpha+1)}{(n-i+\alpha)} p_i\right\} + \ldots + \frac{b_1 \ldots b_{i-1}}{a_1 \ldots a_{i-1}}$$

$$\cdot \left\{\frac{n-i+\alpha}{\alpha} p_i + \frac{(1-\alpha)}{\alpha} \frac{(2n-2i+\alpha+1)}{(n-i+\alpha)} p_i\right\} + \frac{b_1 \ldots b_i}{a_1 \ldots a_i^3} \left\{\frac{n-i-1+\alpha}{\alpha} p_{i+1}\right\}$$

$$\left. + \ldots + \frac{b_1 \ldots b_{n-1}}{a_1 \ldots a_{n-1}} p_n\right].$$

Again using equation (4.3.2), we get

$$E(z_i^2) = \frac{4\alpha}{(n-1)} \left[\left\{\frac{n-1}{na_1} + \frac{n-2}{na_1 a_2} + \ldots + \frac{n-i+1}{na_1 \ldots a_{i-1}}\right\} + \frac{(1-\alpha)}{\alpha}\right.$$

$$\cdot \frac{(2n-2i+\alpha+1)}{(n-i+\alpha)} p_i \{\frac{b_1}{a_1} + \frac{b_1 b_2}{a_1 a_2} + \ldots + \frac{b_1 \ldots b_{i-1}}{a_1 \ldots a_{i-1}}\} + \frac{1}{a_1 \ldots a_i^3}$$

$$\cdot \{\frac{n-i}{n} + \frac{n-i-1}{n a_{i+1}} + \ldots + \frac{1}{n a_{i+1} \ldots a_{n-1}}\}]. \qquad (5.3.6)$$

$$= \frac{4\alpha}{n(n-1)} [\frac{\Gamma(n+1)}{\Gamma(n+2\alpha-1)} \{\frac{\Gamma(n+2\alpha-2)}{\Gamma(n-1)} + \frac{\Gamma(n+2\alpha-3)}{\Gamma(n-2)} + \ldots + \frac{\Gamma(n-i+2\alpha)}{\Gamma(n-i+1)}\}$$

$$+ \frac{1-\alpha}{\alpha} \frac{(2n-2i+\alpha+1)}{(n-i+\alpha)} p_i n \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)} \{\frac{\Gamma(n+2\alpha-2)}{\Gamma(n+\alpha-1)} + \ldots +$$

$$\frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i+1)}\} + \frac{1}{a_1 \ldots a_i^3} \frac{\Gamma(n-i+1)}{\Gamma(n+2\alpha-i-1)} \{\frac{\Gamma(n+2\alpha-i-1)}{\Gamma(n-i)} +$$

$$\frac{\Gamma(n+2\alpha-i-2)}{\Gamma(n-i-1)} + \ldots + \frac{\Gamma(2\alpha)}{\Gamma(1)}\}],$$

on substituting the values of a_i's and b_i's. On using equation (4.3.6), it simplifies to

$$E(z_i^2) = \frac{4\alpha}{n(n-1)} [\frac{\Gamma(n+1)}{\Gamma(n+2\alpha-1)} \frac{1}{2\alpha} \{\frac{\Gamma(n+2\alpha-1)}{\Gamma(n-1)} - \frac{\Gamma(n+2\alpha-i)}{\Gamma(n-i)}\} + \frac{1-\alpha}{\alpha}$$

$$\cdot n p_i \frac{(2n-2i+\alpha+1)}{(n-i+\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)} \frac{1}{\alpha} \{\frac{\Gamma(n+2\alpha-1)}{\Gamma(n+\alpha-1)} - \frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i)}\} +$$

$$\frac{1}{a_1 \ldots a_i^3} \frac{\Gamma(n-i+1)}{\Gamma(n+2\alpha-i-1)} (\frac{1}{2\alpha} \{\frac{\Gamma(n+2\alpha-i)}{\Gamma(n-i)} - \frac{\Gamma(2\alpha+1)}{\Gamma(1)}\} + \frac{\Gamma(2\alpha)}{\Gamma(1)})]$$

$$= \frac{4\alpha}{n(n-1)} [\frac{n(n-1)}{2\alpha} - \frac{\Gamma(n+1)\,\Gamma(n+2\alpha-i)}{2\alpha\,\Gamma(n+2\alpha-1)\,\Gamma(n-i)} + (1-\alpha) \frac{n\,\Gamma(n)}{\Gamma(n+\alpha)} \frac{\Gamma(n-i+\alpha)}{\Gamma(n-i+1)}$$

$$\cdot \frac{(2n-2i+\alpha+1)}{(n-i+\alpha)} \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)} \frac{1}{\alpha} \{\frac{\Gamma(n+2\alpha-1)}{\Gamma(n+\alpha-1)} - \frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i)}\} +$$

$$\frac{(n-i)(n+2\alpha-i-1)}{2\alpha} \frac{1}{a_1 \ldots a_i^3}].$$

Hence

$$E(Z_i^2) = \frac{4}{n(n-1)}\left[\frac{n(n-1)}{2} - \frac{\Gamma(n+1)\,\Gamma(n+2\alpha-i)}{2\Gamma(n+2\alpha-1)\,\Gamma(n-i)} + (1-\alpha)\right.$$

$$\cdot\frac{\Gamma(n+1)\,\Gamma(n-i+\alpha)\,(2n-2i+\alpha+1)}{\Gamma(n-i+1)\,\Gamma(n+2\alpha-1)\,(n-i+\alpha)}\left\{\frac{\Gamma(n+2\alpha-1)}{\Gamma(n+\alpha-1)} - \frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i)}\right\} +$$

$$\left.\frac{1}{a_1\cdots a_i^3}\,\frac{(n-i)(n+2\alpha-i-1)}{2}\right]. \tag{5.3.7}$$

For obtaining the marginal pdf of Z_i and Z_j, we integrate out the other variables $Z_1,\ldots,Z_{i-1},Z_{i+1},\ldots,Z_{j-1},Z_{j+1},\ldots,Z_n$ in equation (5.3.2) to get for $i < j$

$$h_{Z_i,Z_j}(z_i,z_j) = h_1(z_i,z_j) + h_2(z_i,z_j) + h_3(z_i,z_j), \quad 0<z_i,z_j<\infty,$$

where

$$h_1(z_i,z_j)$$

$$= \frac{2\alpha^2}{n(n-1)}\left[\left\{\frac{e^{-z_i-z_j}}{a_1b_2} + \frac{e^{-z_i-z_j}}{a_1b_2b_3} +\cdots+ \frac{e^{-z_ib_i-z_j}}{a_1b_2\cdots b_{i-1}} + \frac{e^{-b_iz_i-z_j}}{a_1b_2\cdots b_{i-1}b_{i+1}}\right.\right.$$

$$+\cdots+ \frac{e^{-b_iz_i-z_j}}{a_1b_2\cdots b_{i-1}b_{i+1}\cdots b_{j-1}} + \frac{e^{-b_iz_i-z_jb_j}}{a_1b_2\cdots b_{i-1}b_{i+1}\cdots b_{j-1}} +\cdots+$$

$$\left.\frac{e^{-b_iz_i-b_jz_j}}{a_1b_2\cdots b_n}\right\} + \left\{\frac{e^{-z_i-z_j}}{a_1a_2b_3} + \frac{e^{-z_i-z_j}}{a_1a_2b_3b_4} +\cdots+ \frac{e^{-z_i-z_j}}{a_1a_2b_3\cdots b_{i-1}} +\right.$$

$$\frac{e^{-b_iz_i-z_j}}{a_1a_2b_3\cdots b_{i-1}} +\cdots+ \frac{e^{-b_iz_i-z_j}}{a_1a_2b_3\cdots b_{i-1}b_{i+1}\cdots b_{j-1}} + \frac{e^{-b_iz_i-b_jz_j}}{a_1a_2b_3\cdots b_{j-1}}$$

$$+\cdots+\left.\frac{e^{-b_iz_i-b_jz_j}}{a_1a_2b_3\cdots b_{i-1}b_{i+1}\cdots b_{j-1}b_{j+1}\cdots b_n}\right\} +\cdots+\left\{\frac{e^{-b_iz_i-z_j}}{a_1\cdots a_{i-1}} +\cdots+\right.$$

$$\frac{e^{-b_iz_i-z_j}}{a_1\cdots a_{i-1}b_{i+1}\cdots b_{j-1}} + \frac{e^{-b_iz_i-z_jb_j}}{a_1\cdots a_{i-1}b_{i+1}\cdots b_{j-1}} +\cdots+$$

$$\left.\left.\frac{e^{-b_iz_i-b_jz_j}}{a_1\cdots a_{i-1}b_{i+1}\cdots b_{j-1}b_{j+1}\cdots b_n}\right\}\right],$$

$h_2(z_i,z_j)$

$$= \frac{2\alpha^2}{n(n-1)} [\{\frac{e^{-a_i z_i - z_j}}{a_1 a_2 \cdots a_{i-1}\ b_{i+1}} + \cdots + \frac{e^{-a_i z_i - z_j}}{a_1 a_2 \cdots a_{i-1} b_{i+1} \cdots b_{j-1}} +$$

$$\frac{e^{-a_i z_i - b_j z_j}}{a_1 a_2 \cdots a_{i-1} b_{i+1} \cdots b_{j-1}} + \cdots + \frac{e^{-a_i z_i - b_j z_j}}{a_1 \cdots b_n}\} + \{\frac{e^{-a_i z_i - z_j}}{a_1 \cdots a_{i-1} a_{i+1} b_{i+2}}$$

$$+ \cdots + \frac{e^{-a_i z_i - z_j}}{a_1 a_2 \cdots a_{i-1} a_{i+1} b_{i+2} \cdots b_{j-1}} + \frac{e^{-a_i z_i - b_j z_j}}{a_1 a_2 \cdots a_{i-1} a_{i+1} b_{i+2} \cdots b_{j-1}}$$

$$+ \cdots + \frac{e^{-a_i z_i - b_j z_j}}{a_1 a_2 \cdots a_{i-1} a_{i+1} b_{i+2} \cdots b_n}\} + \cdots + \frac{e^{-a_i z_i - b_j z_j}}{a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1}}$$

$$+ \cdots + \frac{e^{-a_i z_i - b_j z_j}}{a_1 \cdots a_{i-1} a_{i+1} \cdots a_{j-1} b_{j+1} \cdots b_n}\}]$$

and

$h_3(z_i,z_j)$

$$= \frac{2\alpha^2}{n(n-1)} [\{\frac{e^{-a_i z_i - a_j z_j}}{a_1 \cdots a_{j-1} b_{j+1}} + \cdots\} + \cdots + \{\frac{e^{-a_i z_i - a_j z_j}}{a_1 \cdots b_n}\}].$$

Here $h_1(z_i,z_j)$ drops out if $i = 1$, $h_3(z_i,z_j)$ drops out if $j = n$.

In particular, the pdf of Z_1, Z_j is

$h_{Z_1,Z_j}(z_1,z_j)$

$$= c[\{\frac{e^{-a_1 z_1 - z_j}}{b_2} + \cdots + \frac{e^{-a_1 z_1 - z_j}}{b_2 \cdots b_{j-1}} + \frac{e^{-a_1 z_1 - b_j z_j}}{b_2 \cdots b_{j-1}} + \cdots + \frac{e^{-a_1 z_1 - b_j z_j}}{b_2 \cdots b_n}\} +$$

$$\{\frac{e^{-a_1 z_1 - z_j}}{a_2 b_3} + \cdots + \frac{e^{-a_1 z_1 - z_j}}{a_2 b_3 \cdots b_{j-1}} + \frac{e^{-a_1 z_1 - b_j z_j}}{a_2 b_3 \cdots b_{j-1}} + \cdots +$$

$$\frac{e^{-a_1 z_1 - b_j z_j}}{a_2 b_3 \cdots b_{j-1} b_{j+1} \cdots b_n}\} + \cdots + \{\frac{e^{-a_1 z_1 - a_j z_j}}{a_2 \cdots a_{j-1} b_{j+1}} + \cdots\} + \cdots +$$

$$\{\frac{e^{-a_1 z_1 - a_j z_j}}{a_2 \cdots a_{j-1} a_{j+1} \cdots a_{n-1} b_n}\}], \quad \text{where} \quad c = \frac{2\alpha^2}{n(n-1)} .$$

Now

$$E(Z_i Z_j) = \int_0^\infty \int_0^\infty z_i z_j \, h_{Z_i Z_j}(z_i, z_j) \, dz_i \, dz_j .$$

After simple integrations, we get

$$E(Z_i Z_j)$$

$$= \frac{2\alpha^2}{n(n-1)} [\{\frac{1}{a_1 b_2} + \frac{1}{a_1 b_2 b_3} + \cdots + \frac{1}{a_1 b_2 \cdots b_{i-1}} + \frac{1}{a_1 b_2 \cdots b_{i-1} b_i^2} +$$

$$\frac{1}{a_1 b_2 \cdots b_{i-1} b_i^2 b_{i+1}} + \cdots + \frac{1}{a_1 b_2 \cdots b_i^2 b_{i+1} \cdots b_{j-1}} + \frac{1}{a_1 b_2 \cdots b_i^2 \cdots b_j^2}$$

$$+ \cdots + \frac{1}{a_1 b_2 \cdots b_{i-1} b_i^2 \cdots b_{j-1} b_j^2 \cdots b_n}\} + \{\frac{1}{a_1 a_2 b_3} + \frac{1}{a_1 a_2 b_3 b_4} + \cdots +$$

$$\frac{1}{a_1 a_2 b_3 \cdots b_{i-1}} + \frac{1}{a_1 a_2 b_3 \cdots b_{i-1} b_i^2} + \cdots + \frac{1}{a_1 a_2 b_3 \cdots b_i^2 \cdots b_{j-1}} +$$

$$\frac{1}{a_1 a_2 b_3 \cdots b_i^2 \cdots b_j^2} + \cdots + \frac{1}{a_1 a_2 b_3 \cdots b_i^2 \cdots b_j^2 \cdots b_n}\} + \cdots +$$

$$\{\frac{1}{a_1 \cdots a_{i-1} a_i^2 b_{i+1}} + \cdots + \frac{1}{a_1 \cdots a_{i-1} a_i^2 b_{i+1} \cdots b_{j-1}} +$$

$$\frac{1}{a_1 \cdots a_{i-1} a_i^2 b_{i+1} \cdots b_{j-1} b_j^2} + \cdots + \frac{1}{a_1 \cdots a_i^2 b_{i+1} \cdots b_j^2 \cdots b_n}\} +$$

$$\{\frac{1}{a_1 a_2 \cdots a_i^2 a_{i+1} b_{i+2}} + \cdots + \frac{1}{a_1 \cdots a_{i-1} a_i^2 a_{i+1} b_{i+2} \cdots b_{j-1}} +$$

$$\frac{1}{a_1 a_2 \cdots a_i^2 a_{i+1} b_{i+2} b_{j-1} b_j^2} + \cdots + \frac{1}{a_1 \cdots a_{i-1} a_i^2 a_{i+1} b_{i+2} \cdots b_j^2 \cdots b_n}\}$$

$$+ \cdots\cdots +$$

$$\{\frac{1}{a_1\cdots a_i^2\cdots a_j^2 b_{j+1}} +\cdots\} +\cdots+\{\frac{1}{a_1\cdots a_i^2\cdots a_j^2\cdots b_n}\}],$$

which can be written as

$E(z_i z_j)$

$$= \frac{2\alpha^2}{n(n-1)} [\frac{b_1}{a_1} \{\frac{1}{b_1 b_2} + \frac{1}{b_1 b_2 b_3} +\cdots+ \frac{1}{b_1\cdots b_{i-1}} + \frac{1}{b_i} (\frac{1}{b_1\cdots b_i} +\cdots+ \frac{1}{b_1\cdots b_{j-1}}) + \frac{1}{b_i b_j} (\frac{1}{b_1\cdots b_j} +\cdots+ \frac{1}{b_1\cdots b_n})\} + \frac{b_1 b_2}{a_1 a_2} \{\frac{1}{b_1 b_2 b_3}$$

$$+\cdots+ \frac{1}{b_1\cdots b_{i-1}} + \frac{1}{b_i}(\frac{1}{b_1\cdots b_i} +\cdots+ \frac{1}{b_1\cdots b_{j-1}}) + \frac{1}{b_j b_i}(\frac{1}{b_1\cdots b_j}$$

$$+\cdots+ \frac{1}{b_1\cdots b_n})\} +\cdots+ \frac{b_1\cdots b_i}{a_1\cdots a_{i-1} a_i^2} \{\frac{1}{b_1\cdots b_{i+1}} +\cdots+ \frac{1}{b_1\cdots b_{j-1}}$$

$$+ \frac{1}{b_j}(\frac{1}{b_1\cdots b_j} +\cdots+ \frac{1}{b_1\cdots b_n})\} + \frac{b_1\cdots b_{i+1}}{a_1\cdots a_i^2 a_{i+1}} \{\frac{1}{b_1\cdots b_{i+2}} +\cdots+$$

$$\frac{1}{b_1\cdots b_{j-1}} + \frac{1}{b_j}(\frac{1}{b_1\cdots b_j} +\cdots+ \frac{1}{b_1\cdots b_n})\} +\cdots+ \frac{b_1\cdots b_j}{a_1\cdots a_i^2\cdots a_j^2}$$

$$\cdot \{\frac{1}{b_1\cdots b_{j+1}} +\cdots+ \frac{1}{b_1\cdots b_n}\} +\cdots+ \frac{b_1\cdots b_{n-1}}{a_1\cdots a_{n-1}} \frac{1}{b_1\cdots b_n}].$$

For simplifying this expression we use equation (4.3.2). Then

$E(z_i z_j)$

$$= \frac{2\alpha}{(n-1)} [\frac{b_1}{a_1} \{\sum_{k=2}^{i-1} p_k + \frac{1}{b_i} \sum_{k=i}^{j-1} p_k + \frac{1}{b_i b_j} \sum_{k=j}^{n} p_k\} + \frac{b_1 b_2}{a_1 a_2} \{\sum_{k=3}^{i-1} p_k$$

$$+ \frac{1}{b_i} \sum_{k=i}^{j-1} p_k + \frac{1}{b_i b_j} \sum_{k=j}^{n} p_k\} +\cdots+ \frac{b_1\cdots b_{i-1}}{a_1\cdots a_{i-1}} \{\sum_{k=i}^{j-1} p_k \frac{1}{b_i} +$$

$$\frac{1}{b_i b_j} \sum_{k=j}^{n} p_k\} + \frac{b_1\cdots b_i}{a_1\cdots a_i^2} \{\sum_{k=i+1}^{j-1} p_k + \frac{1}{b_j} \sum_{k=j}^{n} p_k\} + \frac{b_1\cdots b_{i+1}}{a_1\cdots a_i^2 a_{i+1}}$$

$$\cdot \{\sum_{k=i+2}^{j-1} p_k + \frac{1}{b_j}\sum_{k=j}^{n} p_k\} + \ldots + \frac{b_1\ldots b_{j-1}}{a_1\ldots a_i^2\ldots a_{j-1}}\{\frac{1}{b_j}\sum_{k=j}^{n} p_k\} +$$

$$\frac{b_1\ldots b_j}{a_1\ldots a_i^2\ldots a_j^2}\{\sum_{k=j+1}^{n} p_k\} + \ldots + \frac{b_1\ldots b_{n-1}}{a_1\ldots a_i^2\ldots a_j^2\ldots a_{n-1}}\{p_n\}]$$

$$= \frac{2\alpha}{(n-1)}[\frac{b_1}{a_1}\{\frac{n-2+\alpha}{\alpha}p_2 - \frac{n-i+\alpha}{\alpha}p_i + \frac{1}{b_i}\frac{n-i+\alpha}{\alpha}p_i - \frac{1}{b_i}\frac{n-j+\alpha}{\alpha}p_j +$$

$$\frac{1}{b_ib_j}\frac{n-j+\alpha}{\alpha}p_j\} + \frac{b_1b_2}{a_1a_2}\{\frac{n-3+\alpha}{\alpha}p_3 - \frac{n-i+\alpha}{\alpha}p_i + \frac{1}{b_i}\frac{n-i+\alpha}{\alpha}p_i -$$

$$\frac{1}{b_i}\frac{n-j+\alpha}{\alpha}p_j + \frac{1}{b_ib_j}\frac{n-j+\alpha}{\alpha}p_j\} + \frac{b_1\ldots b_{i-1}}{a_1\ldots a_{i-1}}\{\frac{1}{b_i}\frac{n-i+\alpha}{\alpha}p_i -$$

$$\frac{1}{b_i}\frac{n-j+\alpha}{\alpha}p_j + \frac{1}{b_ib_j}\frac{n-j+\alpha}{\alpha}p_j\} + \frac{b_1\ldots b_i}{a_1\ldots a_i^2}\{\frac{n-i-1+\alpha}{\alpha}p_{i+1} -$$

$$\frac{n-j+\alpha}{\alpha}p_j + \frac{1}{b_j}\frac{n-j+\alpha}{\alpha}p_j\} + \frac{b_1\ldots b_{i+1}}{a_1\ldots a_i^2a_{i+1}}\{\frac{n-i-2+\alpha}{\alpha}p_{i+2} -$$

$$\frac{n-j+\alpha}{\alpha}p_j + \frac{1}{b_j}\frac{n-j+\alpha}{\alpha}p_j\} + \ldots + \frac{b_1\ldots b_{j-1}}{a_1\ldots a_i^2\ldots a_{j-1}}\{\frac{1}{b_j}\frac{n-j+\alpha}{\alpha}p_j\}$$

$$+ \frac{b_1\ldots b_j}{a_1\ldots a_j^2}\{\frac{n-j-1+\alpha}{\alpha}p_{j+1}\} + \ldots + \frac{b_1\ldots b_{n-1}}{a_1\ldots a_i^2\ldots a_j^2\ldots a_{n-1}}p_n],$$

on using equation (4.3.3). After simplification, we get

$E(Z_iZ_j)$

$$= \frac{2\alpha}{(n-1)}[\frac{b_1}{a_1}\frac{n-2+\alpha}{\alpha}\frac{\alpha}{nb_1b_2} + \frac{b_1b_2}{a_1a_2}\frac{n-3+\alpha}{\alpha}\frac{\alpha}{nb_1b_2b_3} + \frac{b_1b_2b_3}{a_1a_2a_3}\frac{n-4+\alpha}{\alpha}$$

$$\cdot\frac{\alpha}{nb_1\ldots b_4} + \ldots + \frac{b_1\ldots b_{i-1}}{a_1\ldots a_{i-1}}\frac{n-i+\alpha}{\alpha}\frac{\alpha}{nb_1\ldots b_i} + \frac{1-\alpha}{\alpha}(p_i + \frac{p_j}{b_i})$$

$$\cdot\{\frac{b_1}{a_1} + \frac{b_1b_2}{a_1a_2} + \ldots + \frac{b_1\ldots b_{i-1}}{a_1\ldots a_{i-1}}\} + \frac{1}{a_1\ldots a_i^2}\frac{n-i-1+\alpha}{n}\frac{1}{b_{i+1}} +$$

$$\frac{n-i-2+\alpha}{a_1\cdots a_i^2 a_{i+1}}\ \frac{1}{nb_{i+2}} + \cdots + \frac{n-j+\alpha}{a_1\cdots a_i^2\cdots a_{j-1}}\ \frac{1}{nb_j} + \frac{1-\alpha}{\alpha}\ p_j\{\frac{b_1\cdots b_i}{a_1\cdots a_i^2}$$

$$+ \frac{b_1\cdots b_{i+1}}{a_1\cdots a_i^2 a_{i+1}} + \cdots + \frac{b_1\cdots b_{j-1}}{a_1\cdots a_{j-1}}\} + \frac{n-j-1+\alpha}{a_1\cdots a_i^2\cdots a_j^2 nb_{j+1}} + \cdots +$$

$$\frac{\alpha}{a_1\cdots a_{n-1} nb_n}].$$

Substituting the values of b_i's for some terms, it reduces to

$$E(Z_iZ_j)$$

$$= \frac{2\alpha}{(n-1)}\ [\{\frac{n-1}{na_1} + \frac{n-2}{na_1a_2} + \cdots + \frac{n-i+1}{na_1\cdots a_{i-1}} + \frac{n-i}{na_1\cdots a_i^2} + \cdots +$$

$$\frac{n-j+1}{a_1\cdots a_i^2\cdots a_{j-1}n} + \frac{n-j}{na_1\cdots a_i^2\cdots a_j^2} + \cdots + \frac{1}{na_1\cdots a_n}\} + \frac{1-\alpha}{\alpha}(p_i + \frac{p_j}{b_i})$$

$$\cdot\{\frac{b_1}{a_1} + \cdots + \frac{b_1\cdots b_{i-1}}{a_1\cdots a_{i-1}}\} + \frac{1-\alpha}{\alpha}\ p_j\{\frac{b_1\cdots b_i}{a_1\cdots a_i^2} + \frac{b_1\cdots b_{i+1}}{a_1\cdots a_i^2 a_{i+1}} + \cdots +$$

$$\frac{b_1\cdots b_{j-1}}{a_1\cdots a_{j-1}}\}]. \tag{5.3.8}$$

On substituting the values of a_i's and b_i's, we get

$$E(Z_iZ_j)$$

$$= \frac{2\alpha}{(n-1)}\ [\frac{\Gamma(n)}{\Gamma(n+2\alpha-1)}\ \{\sum_{k=1}^{i-1} \frac{\Gamma(n+2\alpha-1-k)}{\Gamma(n-1-k+1)} + \frac{1}{a_i}\sum_{k=1}^{j-i}\frac{\Gamma(n+2\alpha-i-k)}{\Gamma(n-i-k+1)} +$$

$$\frac{1}{a_ia_j}\sum_{k=1}^{n-j-1}\frac{\Gamma(n+2\alpha-j-k)}{\Gamma(n-j-k+1)} + \frac{1}{a_ia_j}\ \frac{\Gamma(2\alpha)}{\Gamma(2)}\} + \frac{1-\alpha}{\alpha}\ p_i\ \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)}\sum_{k=1}^{i-1}$$

$$\cdot\frac{\Gamma(n+2\alpha-1-k)}{\Gamma(n+\alpha-1-k+1)} + \frac{1-\alpha}{\alpha}\ \frac{p_j}{b_i}\ \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)}\sum_{k=1}^{i-1}\frac{\Gamma(n+2\alpha-1-k)}{\Gamma(n+\alpha-1-k+1)} +$$

$$\frac{1-\alpha}{\alpha}\ \frac{p_j}{a_i}\ \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)}\sum_{k=1}^{j-i}\frac{\Gamma(n+2\alpha-i-k)}{\Gamma(n+\alpha-i-k+1)}].$$

Using equation (4.3.6), it gives

$E(z_i z_j)$

$$= \frac{2\alpha}{(n-1)} \Big[\frac{\Gamma(n)}{\Gamma(n+2\alpha-1)} \Big\{ \frac{1}{2\alpha} \Big(\frac{\Gamma(n+2\alpha-1)}{\Gamma(n-1)} - \frac{\Gamma(n+2\alpha-i)}{\Gamma(n-i)} \Big) + \frac{n-i+1}{n-i+2\alpha-1} \, \frac{1}{2\alpha}$$

$$\cdot \Big(\frac{\Gamma(n+2\alpha-i)}{\Gamma(n-i)} - \frac{\Gamma(n+2\alpha-j)}{\Gamma(n-j)} \Big) + \frac{(n-i+1)(n-j+1)}{(n-i+2\alpha-1)(n-j+2\alpha-1)} \Big(\frac{1}{2\alpha} \Big(\frac{\Gamma(n+2\alpha-j)}{\Gamma(n-j)} -$$

$$\frac{\Gamma(2\alpha+1)}{\Gamma(2)} \Big) + \frac{\Gamma(2\alpha)}{\Gamma(2)} \Big) \Big\} + \frac{1-\alpha}{\alpha} \, \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)} \Big(p_i + \frac{p_j}{b_i} \Big) \frac{1}{\alpha} \Big\{ \frac{\Gamma(n+2\alpha-1)}{\Gamma(n+\alpha-1)} -$$

$$\frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i)} \Big\} + \frac{1-\alpha}{\alpha} \, \frac{p_j}{a_i} \, \frac{\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)} \, \frac{1}{\alpha} \Big\{ \frac{\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i)} - \frac{\Gamma(n+2\alpha-j)}{\Gamma(n+\alpha-j)} \Big\} \Big]$$

$$= \frac{2\alpha}{(n-1)} \Big[\frac{n-1}{2\alpha} - \frac{\Gamma(n)\,\Gamma(n+2\alpha-i)}{2\alpha\,\Gamma(n+2\alpha-1)\,\Gamma(n-i)} + \frac{\Gamma(n)\,\Gamma(n+2\alpha-i)\,(n-i+1)}{2\alpha\,\Gamma(n+2\alpha-1)\,\Gamma(n-i)\,(n+2\alpha-i-1)}$$

$$- \frac{\Gamma(n)\,\Gamma(n+2\alpha-j)\,(n-i+1)}{2\alpha(n-i+2\alpha-1)\,\Gamma(n-j)\,\Gamma(n+2\alpha-1)} + \frac{(n-i+1)(n-j+1)\,\Gamma(n)\,\Gamma(n+2\alpha-j)}{(n-i+2\alpha-1)(n-j+2\alpha-1)\,2\alpha\,\Gamma(n-j)}$$

$$\cdot \frac{1}{\Gamma(n+2\alpha-1)} + \frac{1-\alpha}{\alpha^2} \Big(p_i + \frac{p_j}{b_i} \Big) \Big\{ (n+\alpha-1) - \frac{\Gamma(n+\alpha)\,\Gamma(n+2\alpha-i)}{\Gamma(n+\alpha-i)\,\Gamma(n+2\alpha-1)} \Big\} +$$

$$\frac{1-\alpha}{\alpha^2} \, \frac{p_j}{a_i} \Big\{ \frac{\Gamma(n+2\alpha-i)\,\Gamma(n+\alpha)}{\Gamma(n+2\alpha-1)\,\Gamma(n+\alpha-i)} - \frac{\Gamma(n+\alpha)\,\Gamma(n+2\alpha-j)}{\Gamma(n+2\alpha-1)\,\Gamma(n+\alpha-j)} \Big\} \Big].$$

Using equation (4.3.2) and simplifying, we finally get

$E(z_i z_j)$

$$= 1 + \frac{2(1-\alpha)\,\Gamma(n-1)\,\Gamma(n+2\alpha-i-1)}{\Gamma(n+2\alpha-1)\,\Gamma(n-i)} + \frac{2(n-i+1)(1-\alpha)\,\Gamma(n-1)\,\Gamma(n+2\alpha-j-1)}{(n-i+2\alpha-1)\,\Gamma(n-j)\,\Gamma(n+2\alpha-1)}$$

$$+ \frac{2(1-\alpha)\,\Gamma(n-1)\,\Gamma(n+\alpha-i)}{\Gamma(n+\alpha-1)\,\Gamma(n-i+1)} - \frac{2(1-\alpha)\,\Gamma(n-1)\,\Gamma(n+2\alpha-i)}{\Gamma(n-i+1)\,\Gamma(n+2\alpha-1)} +$$

$$\frac{2(1-\alpha)\,\Gamma(n-1)\,\Gamma(n-j+\alpha)\,(n-j+1)}{\Gamma(n+\alpha-1)\,\Gamma(n-j+1)\,(n-j+\alpha)} + \frac{2(1-\alpha)\,\Gamma(n-1)\,\Gamma(n-j+\alpha)\,\Gamma(n+2\alpha-i)}{\Gamma(n-j+1)\,\Gamma(n+\alpha-i)\,\Gamma(n+2\alpha-1)\,(n-j+\alpha)}$$

$$\cdot (n-j+1) + \frac{2(1-\alpha)\,\Gamma(n-1)\,\Gamma(n-j+\alpha)\,\Gamma(n+2\alpha-i)(n-i+1)}{\Gamma(n-j+1)\,\Gamma(n+2\alpha-1)\,\Gamma(n+\alpha-i)\,(n+2\alpha-i-1)} -$$

$$\frac{2(1-\alpha)\,\Gamma(n-1)\,\Gamma(n+2\alpha-j)(n-i+1)}{\Gamma(n-j+1)\,\Gamma(n+2\alpha-1)(n-i+2\alpha-1)} \,. \tag{5.3.9}$$

These formulae given in equations (5.3.5), (5.3.7) and (5.3.9), give the moments of Z_i's. For finding the moments of $Y_{(i)}$'s, we use equation (4.3.1), viz., $Y_{(i)} = \sum_{j=1}^{i} \frac{Z_j}{(n-j+1)}$.

Here for calculation purposes, we have used equations (4.3.1), (5.3.4), (5.3.6) and (5.3.8). In Table 5.3.1 and Table 5.3.2, we have tabulated the expected values of $Y_{(r)}$ for $r = 1,\ldots,n$ and variances and covariances of $Y_{(r)}$ and $Y_{(s)}$ for $1 \le r < s \le n$ for $n = 10$ and $\alpha = .05, .1, .2, .5, 1.0, 2.0, 5.0, 10.0$. These moments are also calculated by the formulae given in Section 5.2. These are also checked by the well known identities

$$\sum_{i=1}^{n} E(Y_{(i)}) = \sum_{i=1}^{n} E(Y_i) = (n - 2 + \frac{2}{\alpha})$$

and $$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{i,j:n} = \sum_{i=1}^{n} V(Y_i) = (n - 2 + \frac{2}{\alpha^2}),$$

where $\sigma_{i,j:n}$ is the covariance between $Y_{(i)}$ and $Y_{(j)}$.

It is to be noted that $\mathrm{Cov}(Y_{(1)},Y_{(j)}) = \mathrm{Var}(Y_{(1)})$, since $Y_{(1)} = Z_1/n$ and $Y_{(j)} = \sum_{i=1}^{n} Z_i/(n-i+1)$, and Z_1 and $(Z_2,\ldots,Z_j)$ are independent as proved in the beginning of this section. However, we have tabulated all covariances in Table 5.3.2 for the sake of completeness.

5.4 The correlation coefficient between the smallest and the largest observations

In this section, we evaluate the correlation coefficient between the smallest and the largest observations. Denoting this correlation coefficient by ρ_2, we have

$$\rho_2^2 = \frac{\mathrm{Cov}^2(Y_{(1)}, Y_{(n)})}{V(Y_{(1)})\ V(Y_{(n)})} .$$

As proved in Section 5.3, Z_1 and $(Z_2, \ldots, Z_n)$ are independent. Hence $Y(1) = \frac{Z_1}{n}$ and $Y_{(j)} - Y_{(1)} = \sum_{i=2}^{j} \frac{Z_i}{n-i+1}$ are independent. Therefore

$$\mathrm{Cov}(Y_{(1)}, Y_{(n)} - Y_{(1)}) = 0,$$

which gives $\mathrm{Cov}(Y_{(1)}, Y_{(n)}) = V(Y_{(1)})$ and

$$\rho_2^2 = \frac{V(Y_{(1)})}{V(Y_{(n)})}, \qquad (5.4.1)$$

where $\rho_2 > 0$ since $\mathrm{Cov}(Y_{(1)}, Y_{(n)}) > 0$. From equations (5.2.3), (5.2.4) and (5.2.6), we also have

$$V(Y_{(1)}) = \frac{1}{(n-2+2\alpha)^2} = \frac{1}{f(\alpha)} \quad \text{(say)} \qquad (5.4.2)$$

and

$$V(Y_{(n)}) = \sum_{i=1}^{n-2} \frac{1}{i^2} + \Big(\sum_{i=1}^{n-2} \frac{1}{i}\Big)^2 - 2B(2\alpha, n-1) \sum_{i=1}^{n-1} \frac{1}{2\alpha+i-1} +$$
$$4B(\alpha, n-1) \sum_{i=1}^{n-1} \frac{1}{\alpha+i-1} - \Big[\sum_{i=1}^{n-2} \frac{1}{i} + 2B(\alpha, n-1) -$$
$$B(2\alpha, n-1)\Big]^2 = g(\alpha) \quad \text{(say)}. \qquad (5.4.3)$$

Let $\rho_2^2 = [h(\alpha)]^{-1}$. Then $h(\alpha) = g(\alpha) f(\alpha)$. On using these equations for $\alpha = 1$, it simplifies to the well known result that

$\rho_2 = [n^2 \sum_{j=1}^{n} \frac{1}{j^2}]^{-1/2}$ (see David (1981), p. 49). Similar expression for correlation coefficient ρ_1 for the case of one outlier, is given by Gross et al. (1986), where it is shown that this correlation coefficient attains a local maximum for $\alpha = 1$. In fact they have shown that there is a global maximum if $\alpha = 1$ for $n \leq 5$. For larger values of n, algebraic expressions become too complicated to prove it theoretically. We now prove a similar result for ρ_2 using an identical approach.

Equations (5.4.2) and (5.1.3) give

$$f(1) = n^2,$$

$$f'(1) = 4n,$$

$$f''(1) = 8,$$

and
$$g(1) = \sum_{i=1}^{n} \frac{1}{i^2} . \tag{5.4.4}$$

Also
$$B'(\alpha,n-1) = \frac{d}{d\alpha} \int_0^1 x^{\alpha-1}(1-x)^{n-2}\, dx.$$

On interchanging the order of differentiation and integration, we have

$$B'(\alpha,n-1) = \int_0^1 \frac{d}{d\alpha} x^{\alpha-1}(1-x)^{n-2}\, dx$$

$$= -B(\alpha,n-1) \sum_{i=1}^{n-1} \frac{1}{\alpha+i-1} , \tag{5.4.5}$$

on using equation (5.1.4). Similarly

$$B'(2\alpha,n-1) = \frac{d}{d\alpha} \int_0^1 x^{2\alpha-1} (1-x)^{n-2}\, dx$$

$$= \int_0^1 \frac{d}{d\alpha} x^{2\alpha-1} (1-x)^{n-2}\, dx$$

$$= - 2B(2\alpha,n-1) \sum_{i=1}^{n-1} \frac{1}{2\alpha+i-1} . \qquad (5.4.6)$$

These give

$$B'(1,n-1) = -\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{i}, \qquad B'(2,n-1) = -\frac{2}{n(n-1)} \sum_{i=2}^{n} \frac{1}{i} .$$

Now $h'(\alpha) = g'(\alpha)f(\alpha) + g(\alpha)f'(\alpha)$.

A necessary and sufficient condition that $h'(1) = 0$ is $g'(1)f(1) + g(1)f'(1) = 0$, that is $-\frac{n}{4}g'(1) = g(1)$, on using equation (5.4.4). Here $g(\alpha)$ satisfies this condition. To this end, note that equation (5.4.3) gives

$$g'(\alpha) = 4B(2\alpha,n-1)[(\sum_{j=1}^{n-1} \frac{1}{2\alpha+j-1})^2 + \sum_{j=1}^{n-1} \frac{1}{(2\alpha+j-1)^2}] -$$

$$4B(\alpha,n-1)[(\sum_{j=1}^{n-1} \frac{1}{\alpha+j-1})^2 + \sum_{j=1}^{n-1} \frac{1}{(\alpha+j-1)^2}] -$$

$$4[\sum_{j=1}^{n-2} \frac{1}{j} + 2B(\alpha,n-1) - B(2\alpha,n-1)][B(2\alpha,n-1) \sum_{j=1}^{n-1} \frac{1}{2\alpha+j-1}$$

$$- B(\alpha,n-1) \sum_{j=1}^{n-1} \frac{1}{\alpha+j-1}],$$

and hence

$$g'(1) = \frac{4}{n(n-1)} [(\sum_{j=1}^{n-1} \frac{1}{j+1})^2 + \sum_{j=1}^{n-1} \frac{1}{(j+1)^2}] - \frac{4}{n-1} [(\sum_{j=1}^{n-1} \frac{1}{j})^2 +$$

$$\sum_{j=1}^{n-1} \frac{1}{(j)^2}] - 2[\sum_{j=1}^{n-2} \frac{1}{j} + \frac{2}{n-1} - \frac{1}{n(n-1)}][\frac{2}{n(n-1)} \sum_{j=1}^{n-1} \frac{1}{j+1} -$$

$$\frac{2}{n-1} \sum_{j=1}^{n-1} \frac{1}{j}]$$

$$= \frac{4}{n-1} [(\sum_{j=2}^{n} \frac{1}{j})^2 + \sum_{j=2}^{n} \frac{1}{j^2} - (\sum_{j=1}^{n-1} \frac{1}{j})^2 - \sum_{j=1}^{n-1} \frac{1}{j^2}] -$$

$$\frac{4}{n}\left[\left(\sum_{j=2}^{n}\frac{1}{j}\right)^2 + \sum_{j=2}^{n}\frac{1}{j^2}\right] - 2\sum_{j=1}^{n}\frac{1}{j}\left[\frac{2}{n-1}\left(\frac{1}{n}-1\right) - \frac{2}{n}\sum_{j=2}^{n}\frac{1}{j}\right].$$

After some simplification, it reduces to

$$g'(1) = -\frac{4}{n}\sum_{j=1}^{n}\frac{1}{j^2} = -\frac{4}{n}\,g(1),$$

on using equation (5.4.4). Further

$$h''(\alpha) = g''(\alpha)f(\alpha) + 2g''(\alpha)f'(\alpha) + g(\alpha)f''(\alpha).$$

Therefore, on using equation (5.4.4), we get

$$h''(1) = n^2 g''(1) + 2\cdot 4n\left(-\frac{4}{n}\right) g(1) + 8g(1)$$

$$= n^2 g''(1) - 24g(1).$$

Differentiating $g'(\alpha)$ with respect to α once and putting $\alpha = 1$, we get

$$g''(1) = -\frac{8}{n(n-1)}\left(\sum_{i=2}^{n}\frac{1}{i}\right)^3 - \frac{16}{n(n-1)}\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2} -$$

$$\frac{8}{n(n-1)}\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2} - \frac{16}{n(n-1)}\sum_{i=2}^{n}\frac{1}{i^3} + \frac{4}{n-1}\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)^3 +$$

$$\frac{8}{n-1}\sum_{i=1}^{n-1}\frac{1}{i}\sum_{i=1}^{n-1}\frac{1}{i^2} + \frac{4}{n-1}\sum_{i=1}^{n-1}\frac{1}{i}\sum_{i=1}^{n-1}\frac{1}{i^2} + \frac{8}{n-1}\sum_{i=1}^{n-1}\frac{1}{i^3} -$$

$$2\left[\frac{2}{n(n-1)}\sum_{i=2}^{n}\frac{1}{i} - \frac{2}{n-1}\sum_{i=1}^{n-1}\frac{1}{i}\right]^2 - 2\left(\sum_{i=1}^{n-2}\frac{1}{i} + \frac{2}{n-1} - \frac{1}{n(n-1)}\right)$$

$$\cdot\left[\frac{2}{n-1}\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)^2 + \frac{2}{n-1}\sum_{i=1}^{n-1}\frac{1}{i^2} - \frac{4}{n(n-1)}\left(\sum_{i=2}^{n}\frac{1}{i}\right)^2 -\right.$$

$$\left.\frac{4}{n(n-1)}\sum_{i=2}^{n}\frac{1}{i^2}\right]$$

$$= -\frac{8n}{(n-1)}\left(\sum_{i=2}^{n}\frac{1}{i}\right)^3 - \frac{16n}{(n-1)}\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2} - \frac{8n}{n-1}\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2}$$

$$- \frac{16n}{n-1} \sum_{i=2}^{n} \frac{1}{i^3} + \frac{4n^2}{n-1} \left(\sum_{i=1}^{n-1} \frac{1}{i} \right)^3 + \frac{8n^2}{n-1} \sum_{i=1}^{n-1} \frac{1}{i} \sum_{i=1}^{n-1} \frac{1}{i^2} +$$

$$\frac{4n^2}{n-1} \sum_{i=1}^{n-1} \frac{1}{i} \sum_{i=1}^{n-1} \frac{1}{i^2} + \frac{8n^2}{n-1} \sum_{i=1}^{n-1} \frac{1}{i^3} - \frac{8n^2}{(n-1)^2} \left(\frac{1}{n} \sum_{i=2}^{n} \frac{1}{i} - \sum_{i=1}^{n-1} \frac{1}{i} \right)^2$$

$$- 2n^2 \left(\sum_{i=1}^{n} \frac{1}{i} \right) \frac{2}{n-1} \left[\left(\sum_{i=1}^{n-1} \frac{1}{i} \right)^2 + \sum_{i=1}^{n-1} \frac{1}{i^2} - \frac{2}{n} \left(\sum_{i=2}^{n} \frac{1}{i} \right)^2 - \right.$$

$$\left. \frac{2}{n} \sum_{i=2}^{n} \frac{1}{i^2} \right]$$

$$= \frac{8n}{n-1} \left(\sum_{i=2}^{n} \frac{1}{i} \right)^2 \left(\sum_{i=1}^{n} \frac{1}{i} - \sum_{i=2}^{n} \frac{1}{i} - \frac{1}{n(n-1)} \right) + \frac{8n}{n-1} \left(n \sum_{i=1}^{n-1} \frac{1}{i} \right.$$

$$\left. \cdot \sum_{i=1}^{n-1} \frac{1}{i^2} - 3 \sum_{i=2}^{n} \frac{1}{i} \sum_{i=2}^{n} \frac{1}{i^2} \right) - \frac{16n}{n-1} \sum_{i=2}^{n} \frac{1}{i^3} - \frac{4n}{n-1} \left(\sum_{i=1}^{n-1} \frac{1}{i} \right)^2 -$$

$$\frac{4n}{n-1} \sum_{i=1}^{n-1} \frac{1}{i^2} + \frac{8n^2}{n-1} \sum_{i=1}^{n-1} \frac{1}{i^3} - \frac{8n^2}{(n-1)^2} \left(\sum_{i=1}^{n-1} \frac{1}{i} \right)^2 +$$

$$\frac{16n}{(n-1)^2} \sum_{i=1}^{n-1} \frac{1}{i} \sum_{i=2}^{n} \frac{1}{i} + \frac{8n}{n-1} \sum_{i=1}^{n} \frac{1}{i} \sum_{i=2}^{n} \frac{1}{i^2}$$

$$= \frac{8n}{n-1} \left(\sum_{i=2}^{n} \frac{1}{i} \right)^2 \left(1 - \frac{1}{n(n-1)} \right) + \frac{8n}{n-1} \left[(n-3) \sum_{i=2}^{n} \frac{1}{i} \sum_{i=2}^{n} \frac{1}{i^2} + \right.$$

$$\left. n \left\{ \frac{n-1}{n} \sum_{i=2}^{n} \frac{1}{i^2} + \frac{n^2-1}{n^2} \sum_{i=2}^{n} \frac{1}{i} + \frac{(n-1)^2(n+1)}{n^3} \right\} \right] +$$

$$\frac{8n}{n-1} \left(n \sum_{i=1}^{n-1} \frac{1}{i^3} - 2 \sum_{i=2}^{n} \frac{1}{i^3} \right) - \frac{4n}{n-1} \left(\sum_{i=1}^{n-1} \frac{1}{i} \right)^2 - \frac{4n}{n-1} \sum_{i=1}^{n-1} \frac{1}{i^2} -$$

$$\frac{8n^2}{(n-1)^2} \left(\sum_{i=1}^{n-1} \frac{1}{i} \right)^2 + \frac{16n}{(n-1)^2} \sum_{i=1}^{n-1} \frac{1}{i} \sum_{i=2}^{n} \frac{1}{i} + \frac{8n}{n-1} \sum_{i=1}^{n} \frac{1}{i} \sum_{i=2}^{n} \frac{1}{i^2} .$$

This gives on some simplification $h''(1)$ as

$$h''(1) = \frac{8n}{n-1}\left(\sum_{i=2}^{n}\frac{1}{i}\right)^2\left(1 - \frac{1}{n(n-1)} - \frac{1}{2}\right) - \frac{4(n-1)}{n} - 8\sum_{i=2}^{n}\frac{1}{i} +$$

$$\frac{8n}{n-1}(n-3)\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2} + 8n\sum_{i=2}^{n}\frac{1}{i^2} + 8(n+1)\sum_{i=2}^{n}\frac{1}{i} +$$

$$\frac{8(n+1)(n-1)}{n} + \frac{8n}{n-1}\left[(n-2)\sum_{i=2}^{n}\frac{1}{i^3} + n - \frac{1}{n^2}\right] +$$

$$\frac{4n}{n-1}\left(\sum_{i=2}^{n}\frac{1}{i^2}\right)\left(2\sum_{i=1}^{n}\frac{1}{i} - 1\right) - \frac{4n}{n-1} + \frac{4}{n(n-1)} - \frac{8n^2}{(n-1)^2}$$

$$\cdot\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)^2 + \frac{16n}{(n-1)^2}\sum_{i=1}^{n-1}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i} - 24\sum_{i=1}^{n}\frac{1}{i^2}$$

$$= \frac{8n}{n-1}\left(\sum_{i=2}^{n}\frac{1}{i}\right)^2\left(1 - \frac{1}{n(n-1)} - \frac{1}{2}\right) + \frac{8n}{n-1}(n-3)\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2} +$$

$$8\left[n\sum_{i=2}^{n}\frac{1}{i^2} + \frac{n^2-1}{n} - 3\sum_{i=1}^{n}\frac{1}{i^2} - \frac{n}{2(n-1)} - \frac{n-1}{2n}\right] + 8n\sum_{i=2}^{n}\frac{1}{i}$$

$$+ \frac{8n}{n-1}\left((n-2)\sum_{i=2}^{n}\frac{1}{i^3} + n - \frac{1}{n^2}\right) + \frac{4n}{n-1}\sum_{i=2}^{n}\frac{1}{i^2}\left(2\sum_{i=1}^{n}\frac{1}{i} - 1\right)$$

$$+ \frac{4}{n(n-1)} - \frac{8n^2}{(n-1)^2}\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)^2 + \frac{16n}{(n-1)^2}\sum_{i=1}^{n-1}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i}$$

$$= \frac{8n}{n-1}\left(\sum_{i=2}^{n}\frac{1}{i}\right)^2\left(\frac{1}{2} - \frac{1}{n(n-1)}\right) + \frac{8n}{n-1}(n-3)\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2} +$$

$$8(n-3)\sum_{i=2}^{n}\frac{1}{i^2} + 8\left[\frac{n^2-1}{n} - 3 - \frac{n}{2(n-1)} - \frac{n-1}{2n} + \frac{1}{2n(n-1)} +\right.$$

$$\left.\frac{n}{n-1}\left\{(n-2)\sum_{i=2}^{n}\frac{1}{i^3} + n - \frac{1}{n^2}\right\}\right] + \frac{8n}{n-1}\left[(n-1)\sum_{i=2}^{n}\frac{1}{i} -\right.$$

$$\left.\frac{n}{n-1}\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)^2 + \frac{2}{n-1}\sum_{i=1}^{n-1}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i}\right] + \frac{4n}{n-1}\sum_{i=2}^{n}\frac{1}{i^2}\left(2\sum_{i=1}^{n}\frac{1}{i} - 1\right),$$

which simplifies to

$$h''(1) = \frac{8n}{n-1}\left(\sum_{i=2}^{n}\frac{1}{i}\right)^2\left(\frac{1}{2}-\frac{1}{n(n-1)}\right) + \frac{8n}{n-1}(n-3)\sum_{i=2}^{n}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i^2} +$$

$$8(n-3)\sum_{i=2}^{n}\frac{1}{i^2} + 8\left[\frac{n^2-1}{n} - 3 - \frac{n}{2(n-1)} - \frac{n-1}{2n} + \frac{1}{2n(n-1)} +\right.$$

$$\frac{n}{n-1}\left\{(n-2)\sum_{i=2}^{n}\frac{1}{i^3} + \frac{n}{2} - \frac{1}{n^2}\right\}\Bigg] + \frac{8n}{n-1}\left[(n-1)\sum_{i=2}^{n}\frac{1}{i} -\right.$$

$$\left.\frac{n}{n-1}\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)^2 + \frac{2}{n-1}\sum_{i=1}^{n-1}\frac{1}{i}\sum_{i=2}^{n}\frac{1}{i} + \frac{n}{2}\right] + \frac{4n}{n-1}\sum_{i=2}^{n}\frac{1}{i^2}$$

$$\cdot\left(2\sum_{i=1}^{n}\frac{1}{i} - 1\right).$$

To complete the proof, we have to show that $h''(1) \geq 0$ for $n \geq 3$. Clearly, the first three terms and the last term are non-negative. We next show that 4th and 5th terms are also positive.

$$\text{Fourth term} = \frac{n^2-1}{n} - 3 - \frac{n}{2(n-1)} - \frac{n-1}{2n} + \frac{1}{2n(n-1)} + \frac{n}{n-1}$$

$$\cdot\left\{(n-2)\sum_{i=2}^{n}\frac{1}{i^3} + \frac{n}{2} - \frac{1}{n^2}\right\},$$

which on simplification gives

$$\text{Fourth term} = \frac{3n^2-11n+6}{2(n-1)} + \frac{n}{n-1}\left\{(n-2)\sum_{j=2}^{n}\frac{1}{j^3} + \frac{1}{2} - \frac{1}{n^2}\right\} + \frac{1}{n(n-1)}$$

$$= \frac{(3n-2)(n-3)}{2(n-1)} + \frac{n}{n-1}\left\{(n-2)\sum_{j=2}^{n}\frac{1}{j^3} + \frac{1}{2} - \frac{1}{n^2}\right\} + \frac{1}{n(n-1)}$$

$$\geq 0 \quad \text{for} \quad n \geq 3.$$

Now

$$\text{Fifth term} = (n-1)\sum_{j=2}^{n}\frac{1}{j} - \frac{n}{n-1}\left(\sum_{j=1}^{n-1}\frac{1}{j}\right)^2 + \frac{2}{n-1}\sum_{j=1}^{n-1}\frac{1}{j}\sum_{j=2}^{n}\frac{1}{j} + \frac{n}{2}$$

$$\geq (n-1) \sum_{j=2}^{n} \frac{1}{j} - \frac{n}{n-1} \left(\sum_{j=1}^{n-1} \frac{1}{j} \right) \frac{n}{2} + \frac{2}{n-1} \sum_{j=1}^{n-1} \frac{1}{j} \sum_{j=2}^{n} \frac{1}{j} + \frac{n}{2}$$

$$= (2(n-1)^2 - n^2) \frac{1}{2(n-1)} \sum_{j=2}^{n} \frac{1}{j} - \frac{n^2}{2(n-1)} \left(1 - \frac{1}{n}\right) + \frac{2}{n-1} \sum_{j=1}^{n-1} \frac{1}{j} \sum_{j=2}^{n} \frac{1}{j} + \frac{n}{2}$$

$$= \sum_{j=2}^{n} \frac{1}{i} \left[\left\{ (n-1)^2 - \frac{n^2}{2} \right\} \frac{1}{n-1} + \frac{2}{n-1} \sum_{j=1}^{n-1} \frac{1}{j} \right]$$

$$\geq \sum_{j=2}^{n} \frac{1}{j} \left[(n-1)^2 - \frac{n^2}{2} + 2 \right] \frac{1}{n-1}$$

$$\geq 0 \quad \text{for } n \geq 3.$$

This shows that $h(\alpha)$ has a minimum at $\alpha = 1$. Consequently $\rho_2^2 = \frac{1}{h(\alpha)}$ and ρ_2 has a local maximum at $\alpha = 1$, since $\rho_2 > 0$.

In Table 5.4.1, we have given values of ρ_2 for $n = 3(1)20$ and $\alpha = .2, .5, 1., 2., 5.$ These table values and some other calculations show that there may be a global maximum at $\alpha = 1$. This is analogous to the conclusion drawn by Gross et al. (1986) for the case of one outlier.

5.5 Maximum likelihood estimation

Using equation (5.1.1) and denoting the likelihood function of the sample by $L(\sigma,\alpha)$ and taking logarithm, we get

$$\log L(\sigma,\alpha) = 2\log \alpha - n \log \sigma - \log n(n-1) - \frac{n\bar{x}}{\sigma} + \log \sum_{i \neq j=1}^{n} e^{(1-\alpha)\left(\frac{x_i + x_j}{\sigma}\right)}. \tag{5.5.1}$$

Differentiating it with respect to α and σ, we get the ml equations as

$$\frac{2\sigma}{\alpha} = \frac{\sum_{i \neq j} (x_i + x_j)\, e^{(1-\alpha)\left(\frac{x_i + x_j}{\sigma}\right)}}{\sum_{i \neq j} e^{(1-\alpha)\left(\frac{x_i + x_j}{\sigma}\right)}} \tag{5.5.2}$$

and
$$n\bar{x} - n\sigma - \frac{(1-\alpha) \sum_{i \neq j} (x_i + x_j)\, e^{(1-\alpha)\left(\frac{x_i + x_j}{\sigma}\right)}}{\sum_{i \neq j} e^{(1-\alpha)(x_i + x_j)/\sigma}} = 0. \tag{5.5.3}$$

On simplification, these become

$$n\bar{x} = \left(n + \frac{2}{\alpha} - 2\right)\sigma \tag{5.5.4}$$

and
$$\sum_{i \neq j} \left(x_i + x_j - \frac{2\sigma}{\alpha}\right) e^{(1-\alpha)\left(\frac{x_i + x_j}{\sigma}\right)} = 0. \tag{5.5.5}$$

Solving equations (5.5.4) and (5.5.5), we get the maximum likelihood estimates of σ and α. But similar to the case of one outlier discussed in Section 4.2, it is not easy to solve these equations simultaneously. We may again use iterative procedure for solving above equations. On substituting the value of α from equation (5.5.4) into equation (5.5.5), we get

$$\sum_{i \neq j} (x_i + x_j - n\bar{x} + (n-2)\sigma)\, e^{\left(\frac{x_i + x_j}{\sigma}\right)\left(1 - \frac{2}{n\bar{x}/\sigma - (n-2)}\right)} = 0, \tag{5.5.6}$$

which is a function of σ only. Let

$$h(\sigma) = \sum_{i \neq j} (x_i + x_j - n\bar{x} + (n-2)\sigma)\, e^{\left(\frac{x_i + x_j}{\sigma}\right)\left(1 - \frac{2}{n\bar{x}/\sigma - (n-2)}\right)}.$$

It can be seen that in this case also $(\sigma, \alpha) = (\bar{x}, 1)$ is one solution of these equations. Whether or not there are other roots, will depend on the observed values of x_i's. Similar to Lemma 4.2.1, we now have the following lemma.

<u>LEMMA 5.5.1</u>: All the roots of equation $h(\sigma) = 0$ must lie in the interval (I_1, I_2), where $I_1 = \dfrac{n\bar{x}-x_{(n-1)}-x_{(n)}}{(n-2)}$ and $I_2 = \dfrac{n\bar{x}-x_{(1)}-x_{(2)}}{(n-2)}$.

<u>Proof</u>: Clearly $h(\sigma) < 0$ if

$$(x_i + x_j - n\bar{x} + (n-2)\sigma) < 0, \quad \forall \ (i,j),\ i \neq j.$$

That is $x_{(n-1)} + x_{(n)} < n\bar{x} - (n-2)\sigma$,

or $$\sigma < \frac{n\bar{x} - x_{(n-1)} - x_{(n)}}{(n-2)} = I_1 .$$

Similarly $h(\sigma) > 0$ if

$$(x_i + x_j - n\bar{x} + (n-2)\sigma) > 0, \quad \forall \ (i,j),\ i \neq j.$$

That is $x_{(1)} + x_{(2)} > n\bar{x} - (n-2)\sigma$,

or $$\sigma > \frac{n\bar{x} - x_{(1)} - x_{(2)}}{(n-2)} = I_2 .$$

Thus $h(\sigma)$ can be zero only if σ is in the interval (I_1, I_2). Note that this interval includes the interval $\left(\dfrac{n\bar{x}-x_{(n)}}{n-1}, \dfrac{n\bar{x}-x_{(1)}}{n-1}\right)$ obtained in Lemma 4.2.1.

We next prove the following result:

<u>RESULT</u>: (i) For the case $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, $h(\sigma)$ is an increasing function of σ at $\sigma = \bar{x}$.

(ii) For the case $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$, $h(\sigma)$ is a decreasing function of σ at $\sigma = \bar{x}$.

Proof: To prove Results (i) and (ii), we evaluate the derivative of $h(\sigma)$ which is given by

$$h'(\sigma) = \sum_{i\neq j} (n-2) e^{\left(\frac{x_i+x_j}{\sigma}\right)\left(1-\frac{2}{n\bar{x}/\sigma-(n-2)}\right)} + \sum_{i\neq j} (x_i+x_j-n\bar{x}+(n-2)\sigma)$$

$$\cdot e^{\left(\frac{x_i+x_j}{\sigma}\right)\left(1-\frac{2}{n\bar{x}/\sigma-(n-2)}\right)} \left\{-\frac{(x_i+x_j)}{\sigma^2}\left(1-\frac{2}{\frac{n\bar{x}}{\sigma}-(n-2)}\right) - \frac{2(x_i+x_j)}{\sigma}\frac{(n\bar{x}-(n-2)\sigma)+(n-2)\sigma}{(n\bar{x}-(n-2)\sigma)^2}\right\}.$$

Therefore

$$h'(\bar{x}) = n(n-1)(n-2) + \sum_{i\neq j} (x_i+x_j-2\bar{x})\left\{-\frac{(x_i+x_j)}{\bar{x}}\frac{2n\bar{x}}{4\bar{x}^2}\right\}$$

$$= n(n-1)(n-2) - \frac{n}{2}\sum_{i\neq j}\frac{(x_i+x_j)^2}{\bar{x}^2} + \frac{n}{\bar{x}}\sum_{i\neq j}(x_i+x_j)$$

$$= n(n-1)(n-2) - \frac{n}{2\bar{x}^2}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i+x_j)^2 - 4\sum_{i=1}^{n}x_i^2\right] + \frac{n}{\bar{x}}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i+x_j) - 2\sum_{i=1}^{n}x_i\right]$$

$$= n(n-1)(n-2) - \frac{n}{2\bar{x}^2}\left[\sum_{i=1}^{n}x_i^2\sum_{j=1}^{n}1 + \sum_{i=1}^{n}1\sum_{j=1}^{n}x_j^2 + 2\sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j - 4\sum_{i=1}^{n}x_i^2\right] + \frac{n}{\bar{x}}\left[2n^2\bar{x} - 2n\bar{x}\right]$$

$$= n(n-1)(n-2) - \frac{n}{2\bar{x}^2}\left[2(n-2)\sum_{i=1}^{n}x_i^2 + 2n^2\bar{x}^2\right] + 2n^2(n-1)$$

$$= \frac{n}{2\bar{x}^2}\left[2(n-1)(n-2)\bar{x}^2 - 2(n-2)\sum_{i=1}^{n}x_i^2 - 2n^2\bar{x}^2 + 4n(n-1)\bar{x}^2\right]$$

$$= \frac{n}{2\bar{x}^2} \left[2(n-1)\bar{x}^2(n-2+2n) - 2n^2\bar{x}^2 - 2(n-2) \sum_{i=1}^{n} x_i^2\right]$$

$$= \frac{2n(n-2)}{2\bar{x}^2} \left[(2n-1)\, \bar{x}^2 - \sum_{i=1}^{n} x_i^2\right].$$

Now if $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, then $h'(\bar{x})$ is positive, i.e., $h(\sigma)$ an increasing function of σ at $\bar{x}$ and if $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$, t $h'(\bar{x})$ is negative, i.e., $h(\sigma)$ is a decreasing function of σ $\bar{x}$. This completes the proof of Results (i) and (ii). This result is also analogous to the one outlier case obtained in Section 4.2.

Thus if $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, there is possibly only on root $\bar{x}$ of the equation $h(\sigma) = 0$ in the interval (I_1, I_2), whi for $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$, there are at least three roots of the equation $h(\sigma) = 0$ in (I_1, I_2). This follows by using an argu ment similar to the one outlier case. We next show that for the case $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, the log likelihood function, $\log L(\sigma,\alpha)$, attains a local maximum at $(\bar{x},1)$.

<u>THEOREM 5.5.1</u>:. For the case $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, likelihood function attains a local maximum at $(\bar{x},1)$.

<u>Proof</u>: The second partial derivatives of log likelihood function given at equation (5.5.1), are

$$\frac{\partial^2}{\partial\alpha^2} \log L(\sigma,\alpha)$$

$$= \frac{\partial}{\partial\alpha} \left[\frac{2}{\alpha} - \frac{\sum_{i\neq j} \left(\frac{x_i+x_j}{\sigma}\right) e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}{\sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}\right]$$

$$= -\frac{2}{\alpha^2} + \frac{\sum_{i\neq j} \left(\frac{x_i+x_j}{\sigma}\right)^2 e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}{\sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}} - \frac{[\sum_{i\neq j} \left(\frac{x_i+x_j}{\sigma}\right) e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}{\left(\sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}\right)^2}$$

$$\frac{\partial^2}{\partial\sigma\partial\alpha} \log L(\sigma,\alpha)$$

$$= \frac{\sum_{i\neq j} \left[\left(\frac{x_i+x_j}{\sigma^2}\right) + (x_i+x_j)^2 \frac{(1-\alpha)}{\sigma^3}\right] e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}{\sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}} +$$

$$\frac{\sum_{i\neq j} \left(\frac{x_i+x_j}{\sigma}\right) e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)} \frac{\partial}{\partial\sigma} \sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}{\left(\sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}\right)^2}$$

and

$$\frac{\partial^2}{\partial\sigma^2} \log L(\sigma,\alpha)$$

$$= \frac{\partial}{\partial\sigma}\left[-\frac{n}{\sigma} + \frac{n\bar{x}}{\sigma^2} - \frac{(1-\alpha)}{\sigma^2} \frac{\sum_{i\neq j} (x_i+x_j)\, e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}{\sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}\right]$$

$$= \frac{n}{\sigma^2} - \frac{2n\bar{x}}{\sigma^3} - \frac{(1-\alpha)}{\sigma^2} \frac{\partial}{\partial\sigma}\left[\frac{\sum_{i\neq j} (x_i+x_j)\, e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}{\sum_{i\neq j} e^{(1-\alpha)\left(\frac{x_i+x_j}{\sigma}\right)}}\right]$$

At $(\bar{x},1)$, these reduce to

$$\left.\frac{\partial^2}{\partial\alpha^2} \log L(\sigma,\alpha)\right|_{(\bar{x},1)} = -6 + \frac{2}{n(n-1)\bar{x}^2}\left[(n-2) \sum_{i=1}^{n} x_i^2 + n^2\bar{x}^2\right],$$

$$\left.\frac{\partial^2}{\partial\sigma\partial\alpha}\log L(\sigma,\alpha)\right|_{(\bar{x},1)} = \frac{2}{\bar{x}}$$

and

$$\left.\frac{\partial^2}{\partial\sigma^2}\log L(\sigma,\alpha)\right|_{(\bar{x},1)} = -\frac{n}{\bar{x}^2}\ .$$

Let $\underset{\sim}{H}$ be a matrix formed by partial second derivatives of $\log L(\sigma,\alpha)$ at $(\bar{x},1)$. Then

$$\underset{\sim}{H} = \begin{bmatrix} -6 + \frac{2}{n(n-1)\bar{x}^2}\left[(n-2)\sum_{i=1}^{n} x_i^2 + n^2\bar{x}^2\right] & \frac{2}{\bar{x}} \\ \frac{2}{\bar{x}} & -\frac{n}{\bar{x}^2} \end{bmatrix}$$

and

$$|\underset{\sim}{H}| = \frac{6n}{\bar{x}^2} - \frac{2}{(n-1)\bar{x}^4}\left[(n-2)\sum_{i=1}^{n} x_i^2 + n^2\bar{x}^2\right] - \frac{4}{\bar{x}^2}$$

$$= \frac{1}{\bar{x}^4(n-1)}\left[2(3n-2)(n-1)\bar{x}^2 - 2(n-2)\sum_{i=1}^{n} x_i^2 - 2n^2\bar{x}^2\right]$$

$$= \frac{2(n-2)}{\bar{x}^4(n-1)}\left[(2n-1)\bar{x}^2 - \sum_{i=1}^{n} x_i^2\right].$$

Clearly $|\underset{\sim}{H}|$ is positive if $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$. Next

$$H_{11} = -6 + \frac{2}{n(n-1)\bar{x}^2}\left[(n-2)\sum_{i=1}^{n} x_i^2 + n^2\bar{x}^2\right]$$

$$< -6 + \frac{2}{n(n-1)\bar{x}^2}\left[(n-2)(2n-1)\bar{x}^2 + n^2\bar{x}^2\right]$$

$$= -6 + \frac{2(n-1)(3n-1)}{n(n-1)}$$

$$= -\frac{2}{n}\ ,$$

i.e., H_{11} is negative. Hence H is a negative definite matrix, which shows that the likelihood function attains a local maximum at $(\bar{x},1)$.

In case (ii) when $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$, the likelihood function has a local minimum at $(\bar{x},1)$ along the plane given at equation (5.5.4). This can be proved as follows:

Substituting the value of α from equation (5.5.4) into equation (5.5.1), we have log of likelihood function conditionally under the conditions (5.5.4) as

$$\log L\left(\sigma, \frac{2}{\frac{n\bar{x}}{\sigma} - (n-2)}\right)$$

$$= L_1(\sigma) = 2\log \frac{2}{\sum_{i=1}^{n} \frac{x_i}{\sigma} - (n-2)} - n \log \sigma - \log n(n-1) - \frac{\sum_{i=1}^{n} x_i}{\sigma} +$$

$$\log \sum_{i\neq j} e^{(x_i+x_j)[1-2/(\sum_{i=1}^{n} \frac{x_i}{\sigma} - (n-2)]/\sigma} .$$

Writing $\frac{n\bar{x}}{\sigma} - (n-2)$ as $g(\sigma)$, we have

$$L_1(\sigma) = 2\log 2 - 2\log g(\sigma) - n \log\sigma - \log n(n-1) - \frac{n\bar{x}}{\sigma} +$$

$$\log \sum_{i\neq j} e^{(x_i+x_j)(1 - \frac{2}{g(\sigma)})/\sigma} .$$

Differentiating it with respect to σ, it gives

$$\frac{\partial}{\partial\sigma} L_1(\sigma) = -\frac{2g'(\sigma)}{g(\sigma)} - \frac{n}{\sigma} + \frac{n\bar{x}}{\sigma^2} +$$

$$\frac{\sum_{i\neq j} \{-(\frac{x_i+x_j}{\sigma^2})(1-\frac{2}{g(\sigma)}) + (\frac{x_i+x_j}{\sigma})(\frac{2g'(\sigma)}{g^2(\sigma)})\} e^{(\frac{x_i+x_j}{\sigma})(1-\frac{2}{g(\sigma)})}}{\sum_{i\neq j} e^{(\frac{x_i+x_j}{\sigma})(1 - \frac{2}{g(\sigma)})}}$$

Second derivative of $L_1(\sigma)$ is given by

$$\frac{\partial^2}{\partial\sigma^2} L_1(\sigma) = \frac{2(g'(\sigma))^2}{g^2(\sigma)} - \frac{2g''(\sigma)}{g(\sigma)} + \frac{n}{\sigma^2} - \frac{2n\bar{x}}{\sigma^3} -$$

$$\frac{\left[\sum_{i\neq j}\left\{\left(\frac{x_i+x_j}{\sigma}\right)\left(\frac{2g'(\sigma)}{g^2(\sigma)}\right)-\left(\frac{x_i+x_j}{\sigma^2}\right)\left(1-\frac{2}{g(\sigma)}\right)\right\}e^{\left(\frac{x_i+x_j}{\sigma}\right)\left(1-\frac{2}{g(\sigma)}\right)}\right]^2}{\left[\sum_{i\neq j} e^{(x_i+x_j(1-\frac{2}{g(\sigma)}))/\sigma}\right]^2}$$

$$+\frac{\sum_{i\neq j} e^{\left(\frac{x_i+x_j}{\sigma}\right)\left(1-\frac{2}{g(\sigma)}\right)}\left\{\frac{x_i+x_j}{\sigma}\,\frac{2g'(\sigma)}{g^2(\sigma)}-\left(\frac{x_i+x_j}{\sigma^2}\right)\left(1-\frac{2}{g(\sigma)}\right)\right\}^2}{\sum_{i\neq j} e^{\left(\frac{x_i+x_j}{\sigma}\right)\left(1-\frac{2}{g(\sigma)}\right)}}$$

$$+\left[\frac{\sum_{i\neq j}(x_i+x_j)e^{\left(1-\frac{2}{g(\sigma)}\right)\left(\frac{x_i+x_j}{\sigma}\right)}\left\{-\frac{1}{\sigma^2}\frac{4g'(\sigma)}{g^2(\sigma)}+\frac{2}{\sigma}\frac{g^2(\sigma)g''(\sigma)-g'(\sigma)2g(\sigma)g'(\sigma)}{g^4(\sigma)}+\frac{2}{\sigma^3}\left(1-\frac{2}{g(\sigma)}\right)\right\}}{\sum_{i\neq j} e^{\left(\frac{x_i+x_j}{\sigma}\right)\left(1-\frac{2}{g(\sigma)}\right)}}\right]$$

It is easy to see that $g(\bar{x}) = 2$, $g'(\bar{x}) = -\frac{n}{\bar{x}}$, $g''(\bar{x}) = \frac{2n}{\bar{x}^2}$, and

hence

$$\left.\frac{\partial^2}{\partial\sigma^2} L_1(\sigma)\right|_{\bar{x}}$$

$$= \frac{2n^2}{\bar{x}^2}\frac{1}{4} - 2\,\frac{2n}{\bar{x}^2}\frac{1}{2} + \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} - \frac{\left(-\sum_{i\neq j}\left(\frac{x_i+x_j}{\bar{x}}\right)\frac{2n}{4\bar{x}}\right)^2}{\left(\sum_{i\neq j} 1\right)^2} +$$

$$+\frac{\sum_{i\neq j}\left\{\frac{x_i+x_j}{\bar{x}}\left(-\frac{2n}{4\bar{x}}\right)\right\}^2}{\sum_{i\neq j} 1}+$$

$$\frac{\sum_{i\neq j}\left\{-\frac{1}{\bar{x}^2}\,2\left(-\frac{n}{\bar{x}}\right)\frac{1}{4}+\frac{2}{\bar{x}}\left(\frac{\frac{8n}{\bar{x}}-\frac{4n^2}{\bar{x}^2}}{16}\right)+\frac{1}{\bar{x}^2}\,\frac{2}{4}\,\frac{n}{\bar{x}}\right\}(x_i+x_j)}{\sum_{i\neq j} 1}$$

$$=\frac{n^2}{2\bar{x}^2}-\frac{2n}{\bar{x}^2}+\frac{n}{\bar{x}^2}-\frac{2n}{\bar{x}^2}-\frac{n^2}{4\bar{x}^2}\left(\frac{\sum_{i\neq j}(x_i+x_j)}{\sum_{i\neq j} 1}\right)^2+\frac{n^2}{4\bar{x}^4}\,\frac{\sum_{i\neq j}(x_i+x_j)^2}{\sum_{i\neq j} 1}+$$

$$\frac{\sum_{i\neq j}\{n+2n-n^2+n\}(x_i+x_j)}{2\bar{x}^3\sum_{i\neq j} 1}\ .$$

On simplification, it reduces to

$$\left.\frac{\partial^2}{\partial\sigma^2}L_1(\sigma)\right|_{\bar{x}}$$

$$=\frac{n^2}{2\bar{x}^2}-\frac{2n}{\bar{x}^2}+\frac{n}{\bar{x}^2}-\frac{2n}{\bar{x}^2}-\frac{n^2}{4\bar{x}^4}\,\frac{1}{n^2(n-1)^2}\left[\sum_{i\neq j}(x_i+x_j)\right]^2+$$

$$\frac{n^2}{4\bar{x}^4}\,\frac{1}{n(n-1)}\sum_{i\neq j}(x_i+x_j)^2+\frac{\sum_{i\neq j}(x_i+x_j)}{\sum_{i\neq j} 1}\left(\frac{2n}{\bar{x}^3}-\frac{n^2}{2\bar{x}^3}\right). \qquad (5.5.9)$$

But as shown earlier

$$\sum_{i\neq j}(x_i+x_j)=2n(n-1)\bar{x}$$

and $$\sum_{i\neq j}(x_i+x_j)^2=2(n-2)\sum_{i=1}^{n}x_i^2+2n^2\bar{x}^2\ .$$

Substituting these values in equation (5.5.9), we have

$$\frac{\partial^2}{\partial\sigma^2} L_1(\sigma)\bigg|_{\bar{x}} = \frac{n^2}{2\bar{x}^2} - \frac{2n}{\bar{x}^2} - \frac{n}{\bar{x}^2} - \frac{1}{4\bar{x}^4(n-1)^2} 4n^2\bar{x}^2(n-1)^2 + \frac{n}{4(n-1)\bar{x}^4}$$

$$\cdot [2(n-2) \sum_{i=1}^{n} x_i^2 + 2n^2\bar{x}^2] + \frac{2n\bar{x}(n-1)}{n(n-1)} \frac{2n}{\bar{x}^3} - \frac{n^2}{\bar{x}^2}$$

$$= - \frac{n(n-2)(2n-1)}{2(n-1)\bar{x}^2} + \frac{n(n-2)}{2\bar{x}^4(n-1)} \sum_{i=1}^{n} x_i^2$$

$$= \frac{n(n-2)}{2(n-1)\bar{x}^4} \{ \sum_{i=1}^{n} x_i^2 - (2n-1)\bar{x}^2 \} ,$$

which is positive for $\sum_{i=1}^{n} x_i^2 > (2n-1)\bar{x}^2$. Hence the likelihood function has a local minimum at $(\bar{x},1)$ along the plane given at equation (5.5.4).

5.6 Comparison of various estimators

We now compare some of the more efficient estimators which have been studied in Chapter 4. We have calculated the exact mse of various estimators and the simulated bias and mse for the case of two outliers. The exact calculations are done by using the expressions given at equations (4.3.4), (5.3.4), (5.3.6) and (5.3.8) used in tabulating the moments of order statistics in Tables 5.3.1 and 5.3.2. It is easy to show that

$$E(U_1) = E(\frac{1}{n} \sum_{i=1}^{n} X_i) = \frac{1}{n} \sum_{i=1}^{n} E(X_{(i)}) = \frac{1}{n} (n - 2 + \frac{2}{\alpha});$$

$$V(U_1) = V(\frac{1}{n} \sum_{i=1}^{n} X_i) = \frac{1}{n^2} V(\sum_{i=1}^{n} X_i) = \frac{1}{n^2} (n - 2 + \frac{2}{\alpha^2});$$

$$\text{mse}(U_1) = \frac{1}{n^2} (n - 2 + \frac{2}{\alpha^2}) + [\frac{1}{n} (n - 2 + \frac{2}{\alpha}) - 1]^2 \quad \text{and}$$

$$\text{mse}(U_2) = \frac{1}{(n+1)^2}\left(n - 2 + \frac{2}{\alpha^2}\right) + \left[\frac{1}{n+1}\left(n - 2 + \frac{2}{\alpha}\right) - 1\right]^2 .$$

The exact values of mse's of U_1, U_2, U_3, U_4, U_7, U_8, U_{10}, U_{11}, U_{12}, U_{13} have been calculated. These are tabulated for relatively efficient estimators, viz., U_1, U_2, U_3, U_4, U_8, U_{11} and U_{13} in Table 5.6.1 for $n = 10$, 20 and $\alpha = .05$, $.10(.10)1.0$, 2.0, 5.0, 10.0. Table 5.6.1 for mse reveals several important features: (i) Estimators like U_1, U_2, U_3, U_4, U_{11} which performed well in the one outlier case continue to perform well in the two outlier case as well. (ii) For very small values of α (≤ 0.30), now U_8 which excludes two largest observations and U_{13} which excludes two largest and two smallest order statistics and, are more suited for this case, do perform better. However, even now U_4 is not far behind. Thus for example, for $n = 10$ and $\alpha = .2$, the efficiency of U_4 relative to U_8 and U_{13} is .6001 and .7880 respectively. The relative efficiency values for $n = 10$ and $\alpha = .3$ are .9600 and 1.0323 respectively. Similar results hold for $n = 20$ also.

Based on some additional calculations for $n = 10$, we find that U_4 is best for $0.35 \leq \alpha \leq 0.55$, while U_3 is best for $0.60 \leq \alpha \leq 0.85$. Thereafter U_2 is best for $0.90 \leq \alpha \leq 1.32$ and U_1 is best for $\alpha \geq 1.33$. Similar conclusions are valid for $n = 20$ with minor changes in the range of α values. These are identical to the conclusions drawn in the one outlier case. We therefore concentrate on U_1, U_2, U_3, U_4, U_{11}, and study their performances relative to U_9, mle and other estimators by simulation. The simulated values of biases and mse's of these estimators along with those of U_9, mmle, T(1), T(2) and T(3)

are given in Table 5.6.2 and Table 5.6.3 for n = 10 and n = 20 using 1000 iterations and 500 iterations respectively. It can be seen that simulated values agree with the exact values to a considerable extent for estimators for which exact values are available. These values show that the estimators which are preferable in one outlier case continue to perform well in two outlier case also. Here mle and mmle also perform better from the biases and mse's point of view for large values of n. Note that U_4 continues to be quite good from mse's and biases consideration for small values of α (≤ 0.5).

Thus, if one suspects that there are two outliers with smaller α values, then one may decide to use U_4. For large suspected α values, one may use any one of U_1, U_2, or mmle. As seen in Sections 4.6 and 4.7, for one outlier case, one may use U_4, U_{11} or T(3). The simulation study show that now T(3) is not that good compared to U_4 for small α values. Thus for $\alpha = 0.5$, the simulated value of the efficiency of T(3) relative to U_4 is 0.7281 for n = 10 and 0.8603 for n = 20. Similarly the exact values of mse show that U_4 is better than U_{11} for $\alpha \leq 0.5$. For larger α values, U_{11} is better than U_4. Keeping everything in mind our final recommendation goes for using U_4 as an estimator of σ, since it is quite robust in the presence of upto two outliers with larger mean.

r \ α	0.05	0.10	0.20	0.50	1.00	2.00	5.00	10.00
1	0.1235	0.1220	0.1191	0.1111	0.1000	0.0833	0.0556	0.0357
2	0.2641	0.2605	0.2536	0.2353	0.2111	0.1773	0.1244	0.0860
3	0.4275	0.4207	0.4081	0.3760	0.3361	0.2846	0.2129	0.1696
4	0.6222	0.6107	0.5895	0.5381	0.4790	0.4093	0.3263	0.2886
5	0.8633	0.8438	0.8089	0.7294	0.6456	0.5573	0.4707	0.4431
6	1.1795	1.1451	1.0862	0.9621	0.8456	0.7381	0.6552	0.6377
7	1.6387	1.5711	1.4622	1.2590	1.0956	0.9684	0.8950	0.8855
8	2.4816	2.3010	2.0463	1.6672	1.4290	1.2815	1.2224	1.2181
9	10.3751	5.6410	3.4720	2.3169	1.9290	1.7608	1.7194	1.7179
10	30.0246	15.0843	7.7544	3.8049	2.9290	2.7396	2.7182	2.7179

r	s \ α	0.05	0.10	0.20	0.50	1.00	2.00	5.00	10.00
1	1	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	2	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	3	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	4	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	5	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	6	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	7	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	8	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	9	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
1	10	0.0152	0.0149	0.0142	0.0124	0.0100	0.0069	0.0031	0.0013
2	2	0.0350	0.0341	0.0323	0.0278	0.0223	0.0158	0.0080	0.0039
2	3	0.0350	0.0341	0.0323	0.0278	0.0223	0.0158	0.0081	0.0041
2	4	0.0350	0.0341	0.0323	0.0278	0.0223	0.0159	0.0082	0.0042
2	5	0.0351	0.0341	0.0323	0.0278	0.0223	0.0159	0.0083	0.0043
2	6	0.0351	0.0341	0.0324	0.0278	0.0223	0.0159	0.0084	0.0043
2	7	0.0351	0.0342	0.0324	0.0278	0.0223	0.0159	0.0084	0.0043
2	8	0.0352	0.0343	0.0325	0.0279	0.0223	0.0160	0.0085	0.0043
2	9	0.0366	0.0352	0.0329	0.0279	0.0223	0.0160	0.0085	0.0043
2	10	0.0370	0.0358	0.0335	0.0281	0.0223	0.0160	0.0085	0.0043
3	3	0.0618	0.0598	0.0562	0.0476	0.0380	0.0275	0.0169	0.0136
3	4	0.0618	0.0598	0.0563	0.0477	0.0380	0.0275	0.0174	0.0144
3	5	0.0618	0.0599	0.0564	0.0477	0.0380	0.0276	0.0178	0.0149
3	6	0.0619	0.0600	0.0565	0.0477	0.0380	0.0277	0.0181	0.0151

contd...

3	8	0.0623	0.0605	0.0570	0.0479	0.0380	0.0279	0.0184	0.0152
3	9	0.0677	0.0641	0.0586	0.0481	0.0380	0.0280	0.0185	0.0152
3	10	0.0693	0.0663	0.0607	0.0487	0.0380	0.0281	0.0185	0.0152
4	4	0.0998	0.0961	0.0894	0.0741	0.0584	0.0434	0.0323	0.0322
4	5	0.0999	0.0962	0.0896	0.0742	0.0584	0.0436	0.0333	0.0334
4	6	0.1001	0.0965	0.0898	0.0743	0.0584	0.0437	0.0341	0.0339
4	7	0.1003	0.0969	0.0903	0.0744	0.0584	0.0440	0.0346	0.0341
4	8	0.1011	0.0979	0.0912	0.0747	0.0584	0.0442	0.0349	0.0342
4	9	0.1157	0.1073	0.0955	0.0753	0.0584	0.0445	0.0350	0.0342
4	10	0.1199	0.1130	0.1009	0.0767	0.0584	0.0447	0.0351	0.0342
5	5	0.1584	0.1511	0.1384	0.1110	0.0862	0.0661	0.0573	0.0606
5	6	0.1587	0.1516	0.1390	0.1113	0.0862	0.0665	0.0588	0.0616
5	7	0.1594	0.1526	0.1400	0.1117	0.0862	0.0669	0.0598	0.0620
5	8	0.1612	0.1549	0.1421	0.1123	0.0862	0.0674	0.0604	0.0622
5	9	0.1954	0.1768	0.1517	0.1135	0.0862	0.0679	0.0607	0.0622
5	10	0.2051	0.1899	0.1538	0.1164	0.0862	0.0684	0.0609	0.0622
6	6	0.2599	0.2443	0.2179	0.1663	0.1262	0.1003	0.0967	0.1018
6	7	0.2614	0.2465	0.2201	0.1670	0.1262	0.1012	0.0985	0.1025
6	8	0.2656	0.2618	0.2249	0.1683	0.1262	0.1020	0.0995	0.1027
6	9	0.3445	0.3014	0.2459	0.1707	0.1262	0.1029	0.1001	0.1028
6	10	0.3663	0.3304	0.2720	0.1767	0.1262	0.1039	0.1003	0.1028
7	7	0.4784	0.4362	0.3695	0.2575	0.1887	0.1564	0.1601	0.1654
7	8	0.4886	0.4492	0.3804	0.2600	0.1887	0.1580	0.1619	0.1657
7	9	0.6861	0.5699	0.4289	0.2650	0.1887	0.1596	0.1628	0.1658
7	10	0.7385	0.6376	0.4871	0.2773	0.1887	0.1614	0.1631	0.1658
8	8	1.2571	1.0496	0.7709	0.4353	0.2998	0.2603	0.2730	0.2771

contd...

8	9	1.9159	1.4265	0.9064	0.4465	0.2998	0.2634	0.2745	0.2772
8	10	2.0754	1.6198	1.0565	0.4733	0.2998	0.2666	0.2751	0.2772
9	9	94.1920	21.1320	4.4671	0.9220	0.5498	0.5023	0.5249	0.5273
9	10	96.8512	22.7808	5.1522	0.9941	0.5498	0.5086	0.5259	0.5274
10	10	498.6872	123.0062	28.9073	3.7799	1.5498	1.4934	1.5264	1.5274

TABLE 5.4.1: Correlation coefficient between $X_{(1)}$ and $X_{(n)}$

n \ α	0.20	0.50	1.00	2.00	5.00
3	0.1288	0.2308	0.2857	0.2152	0.0944
4	0.0756	0.1576	0.2095	0.1576	0.0756
5	0.0536	0.1206	0.1653	0.1274	0.0663
6	0.0417	0.0981	0.1365	0.1078	0.0600
7	0.0341	0.0829	0.1162	0.0939	0.0552
8	0.0289	0.0719	0.1011	0.0833	0.0512
9	0.0251	0.0637	0.0895	0.0750	0.0479
10	0.0221	0.0572	0.0803	0.0682	0.0450
11	0.0199	0.0519	0.0728	0.0626	0.0424
12	0.0180	0.0476	0.0666	0.0578	0.0402
13	0.0165	0.0440	0.0614	0.0538	0.0382
14	0.0152	0.0409	0.0569	0.0503	0.0363
15	0.0141	0.0382	0.0530	0.0472	0.0347
16	0.0131	0.0359	0.0497	0.0445	0.0332
17	0.0123	0.0338	0.0467	0.0420	0.0318
18	0.0116	0.0320	0.0441	0.0399	0.0306
19	0.0109	0.0304	0.0417	0.0379	0.0294
20	0.0104	0.0289	0.0396	0.0361	0.0283

TABLE 5.6.1: The exact mean square errors of various estimators

n	α	U_1	U_2	U_3	U_4	U_8	U_{11}	U_{13}
10	.05	22.5200	17.9918	14.4737	2.3323	.1183	3.2056	.2277
	.10	5.3200	4.1074	3.2474	.6018	.1159	.8773	.1951
	.20	1.2200	.8843	.6809	.1968	.1181	.2918	.1551
	.30	.5200	.3609	.2771	.1299	.1247	.1791	.1341
	.40	.2950	.2025	.1601	.1104	.1330	.1380	.1232
	.50	.2000	.1405	.1172	.1045	.1418	.1189	.1179
	.60	.1533	.1129	.1001	.1038	.1506	.1093	.1160
	.70	.1282	.1000	.0935	.1057	.1590	.1045	.1161
	.80	.1138	.0940	.0917	.1087	.1671	.1025	.1175
	.90	.1052	.0915	.0922	.1123	.1748	.1022	.1198
	1.0	.1000	.0909	.0937	.1162	.1820	.1028	.1225
	2.0	.0950	.1033	.1143	.1494	.2339	.1219	.1530
	5.0	.1064	.1226	.1383	.1874	.2910	.1524	.1968
	10.0	.1126	.1311	.1486	.2035	.3156	.1658	.2150
20	.05	5.6550	4.9592	4.4124	.5508	.0544	.6684	.0719
	.10	1.3550	1.1497	1.0071	.1623	.0547	.2057	.0653
	.20	.3300	.2653	.2278	.0711	.0576	.0879	.0582
	.30	.1550	.1217	.1049	.0563	.0617	.0648	.0556
	.40	.0988	.0782	.0692	.0525	.0661	.0565	.0551
	.50	.0750	.0612	.0560	.0519	.0704	.0529	.0557
	.60	.0633	.0537	.0507	.0526	.0745	.0514	.0569
	.70	.0570	.0501	.0485	.0539	.0784	.0511	.0584
	.80	.0534	.0485	.0479	.0554	.0819	.0513	.0601
	.90	.0513	.0478	.0479	.0570	.0852	.0518	.0618
	1.0	.0500	.0476	.0483	.0586	.0883	.0525	.0635
	2.0	.0488	.0510	.0540	.0703	.1079	.0602	.0764
	5.0	.0516	.0563	.0606	.0816	.1257	.0692	.0899
	10.0	.0532	.0586	.0634	.0861	.1325	.0729	.0950

TABLE 5.6.2: Simulated values of bias of various estimators based on 1000 iterations for n = 10 and 500 iterations for n = 20 when there are two outliers

Est \ α	.05	.1	.2	.5	1.	2.	5.	10.
$n = 10$								
U_1	3.8392	1.7978	.8223	.2216	.0139	-.0892	-.1504	-.1862
U_2	3.3993	1.5435	.6567	.1106	-.0783	-.1720	-.2276	-.2602
U_3	3.0321	1.3490	.5451	.0467	-.1287	-.2185	-.2740	-.3065
U_4	1.0304	.4155	.1453	-.0733	-.1997	-.2917	-.3608	-.4014
U_9	1.3741	.6698	.3510	.0489	-.1072	-.2025	-.2667	-.3056
U_{11}	1.2683	.5778	.2735	.0279	-.1126	-.2143	-.2878	-.3311
mle	1.6142	.8090	.4775	.1827	.0066	-.1011	-.1616	-.1878
mmle	2.8287	1.4161	.7111	.2157	.0199	-.0876	-.1586	-.2000
T(1)	1.4665	.7027	.3685	.0790	-.0838	-.1819	-.2357	-.2601
T(2)	1.3943	.6078	.2799	.0274	-.1205	-.2158	-.2838	-.3224
T(3)	1.3961	.6112	.2851	.0469	-.0982	-.1980	-.2693	-.3108
$n = 20$								
U_1	1.9898	.9208	.3877	.1172	.0075	-.1622	-.0687	-.0856
U_2	1.8474	.8294	.3217	.0640	-.0405	-.1069	-.1131	-.1292
U_3	1.7336	.7654	.2811	.0353	-.0657	-.1307	-.1376	-.1535
U_4	.4987	.2068	.0457	-.1552	-.1291	-.1975	-.2080	-.2199
U_9	.5434	.3019	.1500	.0349	-.0549	-.1253	-.1332	-.1432
U_{11}	.5791	.2708	.1008	-.0060	-.0834	-.1554	-.1659	-.1780
mle	.5835	.3105	.1828	.0958	.0016	-.0695	-.0784	-.0828
mmle	1.5169	.7612	.3471	.1144	.0089	-.0625	-.0711	-.0878
T(1)	.5341	.2746	.1389	.0452	-.0560	-.1133	-.1209	-.1263
T(2)	.6293	.2937	.1206	.0228	-.0617	-.1342	-.1439	-.1517
T(3)	.6260	.2892	.1261	.0331	-.1486	-.1242	-.1330	-.1434

TABLE 5.6.3: Simulated values of mse of various estimator based on 1000 iterations for n = 10 and 500 iterations for n = 20 when there are two outliers

Est \ α	.05	.1	.2	.5	1.	2.	5.	10.
n = 10								
U_1	22.8915	5.3565	1.2388	.2176	.0976	.0954	.1009	.1144
U_2	18.2923	4.1380	.8961	.1515	.0867	.1019	.1165	.1336
U_3	14.7237	3.2689	.6887	.1243	.0886	.1119	.1318	.1514
U_4	2.1963	.5852	.2009	.0999	.1108	.1439	.1806	.2091
U_9	5.9626	1.4303	.4471	.1283	.0924	.1132	.1366	.1579
U_{11}	3.4200	.8527	.2994	.1168	.0999	.1184	.1458	.1699
mle	7.3367	2.0070	.7141	.2093	.1104	.1109	.1266	.1407
mmle	12.1307	3.2081	.9266	.2061	.1001	.0952	.1016	.1157
T(1)	6.0181	1.5762	.5226	.1502	.0975	.1157	.1367	.1520
T(2)	5.3881	1.2542	.3773	.1304	.0995	.1212	.1491	.1710
T(3)	5.4180	1.2803	.3797	.1372	.1053	.1215	.1457	.1701
n = 20								
U_1	6.3323	1.4775	.3197	.0819	.0579	.0496	.0492	.0529
U_2	5.5653	1.2589	.2570	.0660	.0541	.0529	.0531	.0580
U_3	4.9570	1.1058	.2206	.0596	.0540	.0563	.0571	.0626
U_4	.6358	.1640	.0665	.0508	.0620	.0747	.0795	.0849
U_9	1.2620	.3120	.1151	.0591	.0563	.0575	.0591	.0622
U_{11}	.7666	.2080	.0816	.0529	.0572	.0636	.0677	.0723
mle	.7736	.2584	.1391	.0784	.0616	.0542	.0601	.0582
mmle	3.5323	.9513	.2562	.0796	.0581	.0496	.0496	.0534
T(1)	.7004	.2244	.1139	.0643	.0577	.0575	.0633	.0622
T(2)	1.1474	.2959	.1028	.0591	.0578	.0608	.0634	.0657
T(3)	1.1360	.2744	.1045	.0616	.0583	.0602	.0631	.0657

CHAPTER 6

ESTIMATION PROBLEMS FOR A TRUNCATED EXPONENTIAL DISTRIBUTION

6.1 Introduction

In Chapter 2, we have considered the truncated exponential distribution. Suppose we have n independent observations $X_1,\ldots,X_n$ from the right truncated exponential distribution with density $f(x;\sigma)$. There are two cases depending on the type of truncation. These are

(i) Proportion of truncation (1-P) on the right is fixed and known. The pdf of X with $P_o = -\log(1-P)$ is

$$f(x;\sigma) = \frac{1}{\sigma P} e^{-x/\sigma} \qquad 0 \leq x \leq \sigma P_o$$
$$= 0 \qquad \text{otherwise}. \qquad (6.1.1)$$

(ii) Truncation point x_o on the right is fixed and known. The pdf $f(x;\sigma)$ is now given by

$$f(x;\sigma) = \frac{1}{\sigma(1-e^{-x_o/\sigma})} e^{-x/\sigma}, \qquad 0 \leq x \leq x_o$$
$$= 0 \qquad \text{otherwise}. \qquad (6.1.2)$$

Although the pdf in both the cases looks similar, yet these two cases have to be considered separately as they give rise to different problems. Similar problems for the two types of truncation for normal distribution are discussed in Schneider (1986). Some other types of truncation along with examples are also discussed by him.

Let $X_{(1)},\ldots,X_{(n)}$ be the corresponding order statistics. Then the joint density of $X_{(1)},\ldots,X_{(n)}$ is given by

$$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)},\ldots,x_{(n)}) = \begin{cases} \dfrac{(n)!}{\sigma^n P^n} e^{-\sum_{i=1}^{n} x_{(i)}/\sigma} & 0 \le x_{(1)} \le \cdots \le x_{(n)} \le \sigma P_o \\ 0 & \text{otherwise} \end{cases} \tag{6.1.3}$$

in case (i) and

$$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)},\ldots,x_{(n)})$$

$$= \begin{cases} \dfrac{(n)!}{\sigma^n (1-e^{-x_o/\sigma})^n} e^{-\sum_{i=1}^{n} x_{(i)}/\sigma} & 0 \le x_{(1)} \le \cdots \le x_{(n)} \le x_o \\ 0 & \text{otherwise} \end{cases} \tag{6.1.4}$$

in case (ii).

For case (i), Saleh et al. (1975) gave exact first and second order moments of order statistics, while Joshi (1978) has given some recurrence relations for the moments of order statistics. Many authors have studied this distribution for case (ii) and the appropriate references are given in Section 1.7. We therefore mainly concentrate in case (i) from Section 6.2 to Section 6.7.

We obtain the joint distribution of S and $X_{(n)}$ and some related distributions in Section 6.2. In Sections 6.3 and 6.4, we have discussed the ml estimator of σ and have obtained the exact distribution of mle for n = 2 and 3. In Section 6.5, the constants a and b are obtained such that the mse of T = $a\bar{X} + bX_{(n)}$, a > 0, b > 0, is minimized. We give the exact distribution of $T = a\bar{X} + bX_{(n)}$ for n=2,3 in Section 6.6. In Section 6.7, the estimation of σ is discussed when the sample

contains one outlier. In the last section, we consider the estimation of σ in the presence of a single outlier with some other results for case (ii).

6.2 Joint distribution of $(\Sigma X_{(i)}, X_{(n)})$ when the proportion of truncation on right is known

In this section, the joint distribution of $(S, X_{(n)})$ where $S = \sum_{i=1}^{n} X_i$, is obtained. This is needed for finding the distribution of mle of σ. The distribution of S for case (ii) has been evaluated by Bain and Weeks (1964). Here we follow a similar technique and take $\sigma = 1$ without any loss of generality.

From the definition of conditional pdf, we have

$$f(x_{(1)},\ldots,x_{(n-1)}/X_{(n)} = x_{(n)}) = \frac{f(x_{(1)},\ldots,x_{(n)})}{f(x_{(n)})}$$

$$= \frac{(n-1)!\, e^{-\sum_{j=1}^{n-1} x_{(j)}}}{(1-e^{-x_{(n)}})^{n-1}},$$

$$0 \le x_{(1)} \le \cdots \le x_{(n)} \le P_o, \tag{6.2.1}$$

on using equation (6.1.3) and the fact that the pdf of $X_{(n)}$ is given by

$$f_{X_{(n)}}(x_{(n)}) = \frac{n}{P^n}(1-e^{-x_{(n)}})^{n-1} e^{-x_{(n)}}, \quad 0 \le x_{(n)} \le P_o. \tag{6.2.2}$$

We first obtain the conditional distribution of $S_{n-1} = \sum_{i=1}^{n-1} X_{(i)}$ given $X_{(n)} = x_{(n)}$, and then use it to derive the joint distribution of $(S, X_{(n)})$. For this, consider the

moment generating function of S_{n-1} given $X_{(n)} = x_{(n)}$. This is given by

$$M(t) = E(e^{tS_{n-1}} | X_{(n)} = x_{(n)})$$

$$= \int_0^{x_{(n)}} \cdots \int_0^{x_{(3)}} \int_0^{x_{(2)}} \frac{e^{-\sum_{j=1}^{n-1} x_{(j)}(1-t)} (n-1)!}{(1-e^{-x_{(n)}})^{n-1}} dx_{(1)} \cdots dx_{(n-1)}$$

$$= \frac{(n-1)!}{(1-e^{-x_{(n)}})^{n-1}} \int_0^{x_{(n)}} \cdots \int_0^{x_{(3)}} e^{-\sum_{j=2}^{n-1} x_{(j)}(1-t)} \left\{\frac{e^{-x_{(1)}(1-t)}}{-(1-t)}\right\}_0^{x_{(2)}}$$

$$\cdot\, dx_{(2)} \cdots dx_{(n-1)}$$

$$= \frac{(n-1)!}{(1-e^{-x_{(n)}})^{n-1}} \int_0^{x_{(n)}} \cdots \int_0^{x_{(3)}} e^{-\sum_{j=3}^{n-1} x_{(j)}(1-t)} e^{-x_{(2)}(1-t)}$$

$$\cdot \left(\frac{1-e^{-x_{(2)}(1-t)}}{(1-t)}\right) dx_{(2)} \cdots dx_{(n-1)} .$$

Let $1-e^{-x_{(2)}(1-t)} = y$, which gives $(1-t)e^{-x_{(2)}(1-t)} dx_{(2)} = dy$.

Thus

$$M(t) = \frac{(n-1)!}{(1-e^{-x_{(n)}})^{n-1}} \int_0^{x_{(n)}} \cdots \int_0^{x_{(4)}} e^{-\sum_{j=3}^{n-1} x_{(j)}(1-t)}$$

$$\cdot \int_0^{1-e^{-x_{(3)}(1-t)}} \frac{y}{(1-t)^2} dy\, dx_{(3)} \cdots dx_{(n-1)}$$

$$= \frac{(n-1)!}{(1-t)^2(1-e^{-x_{(n)}})^{n-1}} \int_0^{x_{(n)}} \cdots \int_0^{x_{(4)}} e^{-\sum_{j=3}^{n-1} x_{(j)}(1-t)}$$

$$\cdot \frac{1}{2} \{(1-e^{-x_{(3)}(1-t)})^2\} dx_{(3)} \cdots dx_{(n-1)} .$$

Again substituting $1-e^{-x_{(3)}(1-t)} = y$ and proceeding in this manner, we get

$$M(t) = \left[\frac{1-e^{-x_{(n)}(1-t)}}{(1-t)(1-e^{-x_{(n)}})}\right]^{n-1} .$$

It can also be obtained by using the identity

$$\int_0^{x_{(n)}} \int_0^{x_{(n-1)}} \cdots \int_0^{x_{(2)}} e^{-s_{n-1}} dx_{(1)} \cdots dx_{(n-1)} = \frac{(1-e^{-x_{(n)}})^{n-1}}{(n-1)!} ,$$

and noting that

$$M(t) = \frac{(n-1)!}{(1-e^{-x_{(n)}})^{n-1}} \int_0^{x_{(n)}} \cdots \int_0^{x_{(2)}} e^{-(1-t)s_{n-1}} dx_{(1)} \cdots dx_{(n-1)} .$$

The characteristic function of s_{n-1} given $X_{(n)} = x_{(n)}$ is

$$E(e^{its_{n-1}} | X_{(n)} = x_{(n)}) = \left\{\frac{1-e^{-x_{(n)}(1-it)}}{(1-it)(1-e^{-x_{(n)}})}\right\}^{n-1} ,$$

which is analogous to the characteristic function of S obtained by Bain and Weeks (1964) for $f(x;\sigma)$ given at equation (6.1.2) with n-1 replaced by n and $x_{(n)}$ by x_o. On using the inversion formula, we get the distribution function of $s_{n-1}/X_{(n)} = x_{(n)}$. From this the density function of $s_{n-1}/X_{(n)} = x_{(n)}$ is given by

$$f(s_{n-1}/X_{(n)} = x_{(n)})$$

$$= \begin{cases} \dfrac{e^{-s_{n-1}}}{(n-2)!(1-e^{-x_{(n)}})^{n-1}} \displaystyle\sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (s_{n-1}-kx_{(n)})^{n-2} , \\ \qquad k_o x_{(n)} \leq s_{n-1} \leq (k_o+1)x_{(n)} \text{ where } k_o = 0,1,2,..,(n-2) \\ 0 \qquad \text{otherwise} . \end{cases}$$

Now making the transformation $S = s_{n-1} + X_{(n)}$, we have the conditional pdf of S given $X_{(n)} = x_{(n)}$ as

$f(s/X_{(n)} = x_{(n)})$

$$= \begin{cases} \dfrac{e^{-s+x_{(n)}}}{(n-2)!(1-e^{-x_{(n)}})^{n-1}} \sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (s-(k+1)x_{(n)})^{n-2} \\ \qquad (k_o+1)x_{(n)} \leq s \leq (k_o+2)x_{(n)} \text{ where } k_o=0,1,2,..,(n-2) \\ 0 \qquad \text{otherwise} \end{cases}$$

(6.2.3)

Using equations (6.2.2) and (6.2.3), it gives the joint distribution of S and $X_{(n)}$ as

$$f(s,x_{(n)}) = \begin{cases} \dfrac{ne^{-s}}{(n-2)!P^n} \sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (s-(k+1)x_{(n)})^{n-2} \\ \qquad (k_o+1)x_{(n)} \leq s \leq (k_o+2)x_{(n)} \\ \text{where } k_o = 0,1,2,..,n-2 \text{ and } 0 \leq x_{(n)} \leq P_o \\ 0 \qquad \text{otherwise .} \end{cases}$$

(6.2.4)

The marginal pdf of S can be obtained from equation (6.2.4) as follows:

The joint pdf of $(S,X_{(n)})$ is non-negative in the region shown in Figure 6.2.1.

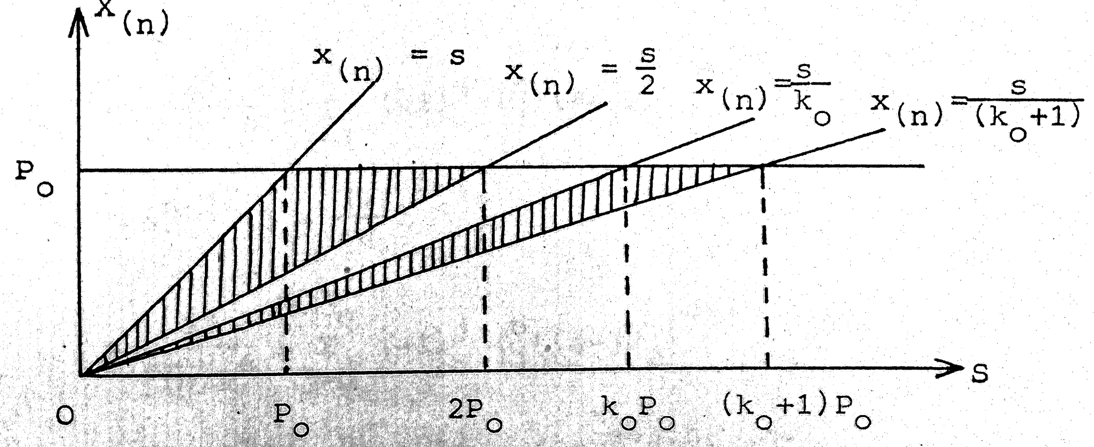

Figure 6.2.1. Showing the region for non-negative density of S and $X_{(n)}$.

For $k_oP_o \leq s \leq (k_o+1)P_o$,

$$f_S(s) = \frac{n\bar{e}^s}{(n-2)!P^n} \Big[\int_{S/(k_o+1)}^{P_o} \sum_{j=o}^{k_o-1} (-1)^j \binom{n-1}{j} (s-(j+1)x_{(n)})^{n-2} \, dx_{(n)} +$$

$$\int_{s/(k_o+2)}^{s/(k_o+1)} \sum_{j=o}^{k_o} (-1)^j \binom{n-1}{j} (s-(j+1)x_{(n)})^{n-2} \, dx_{(n)} + \ldots +$$

$$\int_{s/n}^{s/(n-1)} \sum_{j=o}^{n-2} (-1)^j \binom{n-1}{j} (s-(j+1)x_{(n)})^{n-2} \, dx_{(n)} \Big]$$

$$= \frac{n\bar{e}^s}{(n-2)!P^n} \Big[\sum_{j=o}^{k_o-1} (-1)^j \binom{n-1}{j} \frac{1}{j+1} \frac{1}{n-1} \{ (s - \frac{(j+1)s}{(k_o+1)})^{n-1} -$$

$$(s-(j+1)P_o)^{n-1} \} + \sum_{j=o}^{k_o} (-1)^j \frac{1}{n-1} \binom{n-1}{j} \frac{1}{j+1} \{ (s - \frac{(j+1)s}{(k_o+2)})^{n-1}$$

$$- (s - \frac{s(j+1)}{(k_o+1)})^{n-1} \} + \ldots + \frac{1}{n-1} \sum_{j=o}^{n-2} (-1)^j \binom{n-1}{j} \frac{1}{j+1}$$

$$\cdot \{ (s - \frac{(j+1)s}{n})^{n-1} - (s - \frac{(j+1)s}{n-1})^{n-1} \} \Big]$$

$$= \frac{\bar{e}^s}{\Gamma(n)P^n} \Big[\sum_{j=o}^{k_o-1} (-1)^{j+1} \binom{n}{j+1} (s-(j+1)P_o)^{n-1} + \sum_{j=o}^{n-2} (-1)^j$$

$$\cdot \binom{n}{j+1} s^{n-1} (1 - \frac{j+1}{n})^{n-1} \Big]$$

$$= \frac{\bar{e}^s}{\Gamma(n)P^n} \Big[\sum_{j=1}^{k_o} (-1)^j \binom{n}{j} (s-jP_o)^{n-1} + \sum_{j=o}^{n-2} (-1)^j \binom{n}{j+1}$$

$$\cdot s^{n-1} (1 - \frac{j+1}{n})^{n-1} \Big]$$

$$= \frac{\bar{e}^s}{\Gamma(n)P^n} \Big[\sum_{j=o}^{k_o} (-1)^j \binom{n}{j} (s-jP_o)^{n-1} - s^{n-1} + \frac{s^{n-1}}{n^{n-1}}$$

$$\cdot \{ (-1)^{n-2} \binom{n}{1} 1^{n-1} + (-1)^{n-3} \binom{n}{2} 2^{n-1} + \ldots + (-1)^o \binom{n}{n-1}$$

$$\cdot (n-1)^{n-1} \} \Big]$$

$$= \frac{\bar{e}^{s}}{\Gamma(n)P^n} \left[\sum_{j=o}^{k_o} (-1)^j \binom{n}{j} (s-jP_o)^{n-1} - \frac{s^{n-1}}{n^{n-1}} (-1)^{n-2} \right.$$

$$\cdot \{(-1)^1 \binom{n}{1} 1^{n-1} + (-1)^2 \binom{n}{2} 2^{n-1} + \ldots + (-1)^{n-1} \binom{n}{n-1}$$

$$\cdot (n-1)^{n-1} + (-1)^n \binom{n}{n} n^{n-1}\} .$$

On using an identity (Feller, 1972, p. 65), it gives

$$f(s) = \begin{cases} \dfrac{\bar{e}^{s}}{\Gamma(n)P^n} \displaystyle\sum_{j=o}^{k_o} (-1)^j \binom{n}{j} (s-jP_o)^{n-1} \\ \quad k_oP_o \le s \le (k_o+1)P_o \text{ where } k_o = 0,1,2,\ldots,(n-1) \\ 0 \qquad\qquad \text{otherwise ,} \end{cases} \tag{6.2.5}$$

which is same as the pdf obtained by Bain and Weeks (1964).

The joint density of $\bar{X} = \frac{S}{n}$ and $X_{(n)}$ can be immediately obtained as

$$f(\bar{x},x_{(n)}) = \begin{cases} \dfrac{n^2\bar{e}^{n\bar{x}}}{(n-2)!P^n} \displaystyle\sum_{k=o}^{k_o} (-1)^k \binom{n-1}{k} (n\bar{x}-(k+1)x_{(n)})^{n-2} \\ (k_o+1)x_{(n)} \le n\bar{x} \le (k_o+2)x_{(n)} \\ \text{where } k_o = 0,1,2,\ldots,n-2, \text{ and } 0 \le x_{(n)} \le P_o \\ 0 \qquad\qquad \text{otherwise .} \end{cases} \tag{6.2.6}$$

We next derive the distribution of $U = S/X_{(n)}$. Let $U = S/X_{(n)}$ and $V = X_{(n)}$. The Jacobian of the transformation is $\frac{\partial(U,V)}{\partial(S,X_{(n)})} = \frac{1}{x_{(n)}} = \frac{1}{v}$.

Using equation (6.2.4), we get the joint distribution of U and V as

$$f(u,v) = \begin{cases} \dfrac{n e^{-uv} v}{P^n (n-2)!} \sum\limits_{k=0}^{k_0} (-1)^k \binom{n-1}{k} (uv-(k+1)v)^{n-2} \\ (k_0+1) \le u \le (k_0+2) \text{ where } k_0 = 0,1,\ldots,(n-2) \text{ and } \\ \qquad 0 \le v \le P_0 \\ 0 \qquad \text{otherwise .} \end{cases}$$

Integrating it with respect to v in various ranges, we get the marginal pdf of U. Thus for $1 \le u \le 2$, we have

$$f(u) = \int_0^{P_0} \frac{n\, e^{-uv}\, v(u-1)^{n-2} v^{n-2}}{(n-2)!P^n}\, dv .$$

On substituting uv = t and u dv = dt, it gives

$$f(u) = \frac{n(u-1)^{n-2}}{P^n u^n} \frac{1}{(n-2)!} \int_0^{uP_0} e^{-t}\, t^{n-1}\, dt .$$

This is in terms of an incomplete gamma function. Using equation (1.2.2), it reduces to

$$f(u) = \frac{n(n-1)(u-1)^{n-2}}{P^n u^n} \left[1 - \sum_{k=0}^{n-1} e^{-uP_0} \frac{(uP_0)^k}{(k)!}\right] \quad \text{for} \quad 1 \le u \le 2.$$

Other integrals are also evaluated in this manner. Finally the pdf of U is obtained as

$$f(u) = \begin{cases} \dfrac{n(n-1)}{(1-e^{-P_0})^n} \sum\limits_{k=0}^{k_0-1} \dfrac{(-1)^k \binom{n-1}{k} \{u-(k+1)\}^{n-2}}{u^n} \left[1 - \sum\limits_{j=0}^{n-1} \dfrac{e^{-uP_0}(uP_0)^j}{(j)!}\right] \\ k_0 \le u \le k_0+1 \text{ where } k_0 = 1,2,\ldots,n-1 \\ 0 \qquad \text{otherwise .} \end{cases}$$

Equation (6.2.7) can be used to obtain the distribution of $\frac{X_{(n)}}{S}$ for this truncated case. Here we obtain it only for the untruncated exponential distribution by setting $P_o = \infty$. The pdf of $W = \frac{1}{U} = \frac{X_{(n)}}{S}$ becomes

$$f(w) = \begin{cases} n(n-1) \sum_{k=o}^{k_o-1} (-1)^k \binom{n-1}{k} (1-(k+1)w)^{n-2} \\ \qquad \frac{1}{k_o+1} \leq w \leq \frac{1}{k_o} \text{ where } k_o = 1,2,\ldots,n-1 \\ 0 \qquad \text{otherwise .} \end{cases} \tag{6.2.8}$$

By using direct integration, we get

$$P[W>w_o] = \begin{cases} 1 & \text{if } w_o < 1/n \\ n(1-w_o)^{n-1} - \binom{n}{2}(1-2w_o)^{n-1} + \binom{n}{3}(1-3w_o)^{n-1} + \ldots + \\ (-1)^{i-1}\binom{n}{i}(1-iw_o)^{n-1} \ldots & \text{otherwise ,} \end{cases}$$

where the series continues as long as $(1-iw_o) > 0$. This agrees with the result given in David (1981, p. 100), where alternative methods are given to derive it.

6.3 Maximum likelihood estimation

Using equation (6.1.3), we obtain the likelihood function $L(\sigma) = L(\sigma | x_1, \ldots, x_n)$ as

$$L(\sigma) = \begin{cases} \frac{1}{\sigma^n P^n} e^{-\sum_{i=1}^{n} x_i/\sigma} & 0 \leq x_i \leq \sigma P_o \quad \forall\, i \\ 0 & \text{otherwise .} \end{cases} \tag{6.3.1}$$

$$L(\sigma) = \begin{cases} \dfrac{1}{\sigma^n P^n} e^{-\sum_{i=1}^{n} x_i/\sigma} & \sigma \geq \dfrac{x_i}{P_o} \quad \forall \; i \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \dfrac{1}{\sigma^n P^n} e^{-\frac{n\bar{x}}{\sigma}} & \sigma \geq \dfrac{x_{(n)}}{P_o} \\ 0 & \sigma < \dfrac{x_{(n)}}{P_o} \; . \end{cases}$$

This form immediately gives that $(\bar{X}, X_{(n)})$ is sufficient for σ. Since the range depends on the parameter σ, hence we study the behaviour of $L(\sigma)$ for $\sigma \geq \dfrac{x_{(n)}}{P_o}$. There are two possibilities:

(a) $L(\sigma)$ has a maximum at $x_{(n)}/P_o$. This is illustrated in Figure 6.3.1.

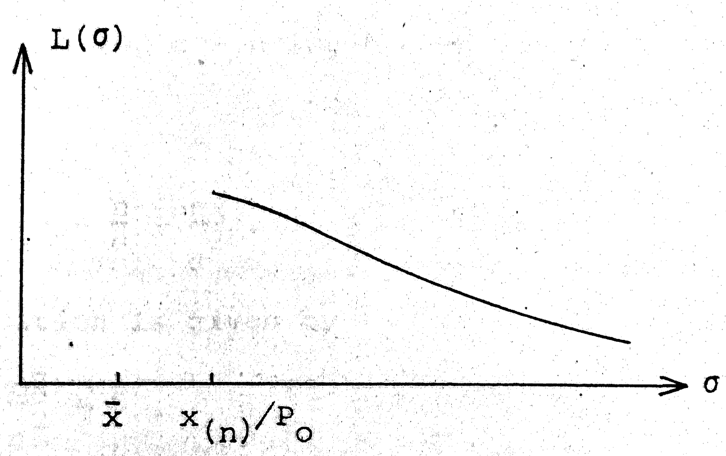

Figure 6.3.1. Showing $L(\sigma)$ plotted against σ for case (a).

(b) $L(\sigma)$ has a maximum at a point greater than $x_{(n)}/P_o$. It is shown in Figure 6.3.2.

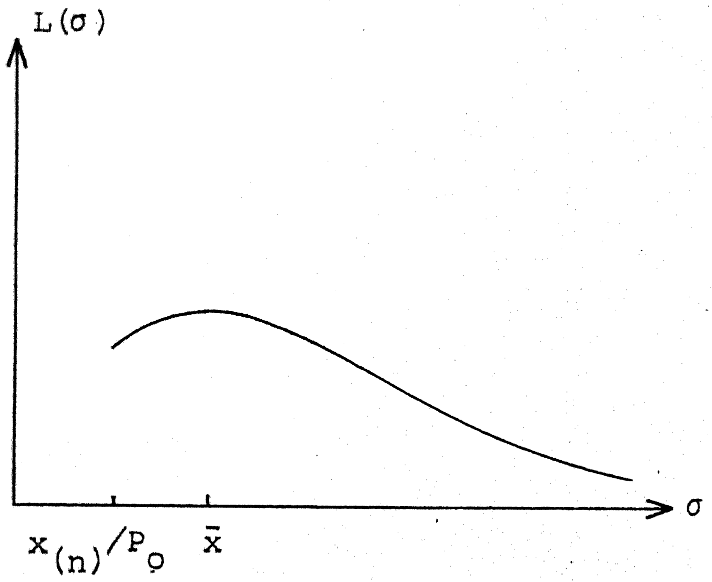

Figure 6.3.2. Showing $L(\sigma)$ plotted against σ for case (b).

Now for $\sigma > \dfrac{x_{(n)}}{P_o}$,

$$\log L(\sigma) = -n \log \sigma - n \log P - \frac{n\bar{x}}{\sigma} .$$

This gives

$$\frac{\partial}{\partial \sigma} \log L(\sigma) = -\frac{n}{\sigma} + \frac{n\bar{x}}{\sigma^2} .$$

Thus ml equation is given by

$$-\frac{n}{\sigma} + \frac{n\bar{x}}{\sigma^2} = 0 ,$$

or $\quad \sigma = \bar{x}$.

We have seen that

$$\frac{\partial}{\partial \sigma} \log L(\sigma) = \frac{n}{\sigma}\left(\frac{\bar{x}}{\sigma} - 1\right)$$

$$= \begin{cases} > 0 & \text{if } \sigma < \bar{x} \\ < 0 & \text{if } \sigma > \bar{x} . \end{cases}$$

This implies that if $\bar{x} < \frac{x_{(n)}}{P_o}$, $L(\sigma)$ is a decreasing function of σ in $(\frac{x_{(n)}}{P_o}, \infty)$ with maximum attained at $\frac{x_{(n)}}{P_o}$. This is the situation shown in Figure 6.3.1. If $\bar{x} > \frac{x_{(n)}}{P_o}$, then $L(\sigma)$ is an increasing function of σ in $(\frac{x_{(n)}}{P_o}, \bar{x})$ and a decreasing function of σ in $(\bar{x}, \infty)$ with maximum attained at $\bar{x}$, since we have $L'(\bar{x}) = 0$ and

$$\frac{\partial^2}{\partial\sigma^2} \log L(\sigma)\Big|_{\bar{x}} = \frac{n}{\bar{x}^2} - \frac{2n}{\bar{x}^2}$$

$$< 0 .$$

This is illustrated in Figure 6.3.2.

Combining these two cases, we get the mle of σ as Z where

$$Z = \max \{\bar{X}, \frac{X_{(n)}}{P_o}\} . \tag{6.3.2}$$

Even though the marginal distributions of $X_{(n)}$ and $n\bar{X}$ have been obtained in equations (6.2.2) and (6.2.5) respectively, yet the distribution of Z is extremely complicated. This can be derived from the joint distribution of $(\bar{X}, X_{(n)})$ given in equation (6.2.6). In the next section we obtain this distribution for some small values of n.

6.4 Exact distribution of mle for small values of n

Denote $Y = X_{(n)}/P_o$. Then the mle $Z = \max(\bar{X}, Y)$. Clearly for $P_o \leq 1$, that is for $P \leq (1-e^{-1})$, we have $Y \geq \bar{X}$ and $Z = \max(\bar{X}, Y) = Y$. Similarly for $P_o \geq n$, that is for $P \geq (1-e^{-n})$, we have

$$Y = \frac{X_{(n)}}{P_o} \le \frac{X_{(n)}}{n} = \frac{\sum_{i=1}^{n} X_i}{n} - \frac{\sum_{i=1}^{n-1} X_i}{n} \le \bar{X} .$$

Hence in this case, $Z = \max(\bar{X}, Y) = \bar{X}$. Since P is known, hence the distribution of mle can be obtained from equation (6.2.2) if $P_o \le 1$ or from equation (6.2.5) if $P_o \ge n$. In particular for n = 1, the distribution of Z is known. For $n \ge 2$, we consider only the cases for which $1 \le P_o \le n$. Now equation (6.2.6) gives the pdf of $\bar{X}$ and Y as

$$f(\bar{x},y) = \begin{cases} \dfrac{n^2 P_o e^{-n\bar{x}}}{(n-2)!P^n} \displaystyle\sum_{k=o}^{k_o-1} (-1)^k \binom{n-1}{k} (n\bar{x} - (k+1)P_o y)^{n-2} & \\ \qquad k_o P_o y \le n\bar{x} \le (k_o+1)P_o y & \\ \qquad \text{where } k_o = 1,2,\ldots,(n-1) \text{ and } 0 \le y \le 1 & \\ 0 \qquad \text{otherwise .} & \end{cases} \tag{6.4.1}$$

Equation (6.4.1) is used to derive the pdf of Z for n = 2 and 3.

<u>Case 1</u>: n = 2. In this case, we only consider $1 \le P_o \le 2$.

Equation (6.4.1) gives the pdf of $(\bar{X}, Y)$ as

$$f(\bar{x},y) = \begin{cases} \dfrac{4P_o e^{-2\bar{x}}}{P^2} & yP_o \le 2\bar{x} \le 2P_o y \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise .} \end{cases}$$

The region of this density is shown in Figure 6.4.1.

We first evaluate $F_Z(z)$, the cdf of Z. This has to be

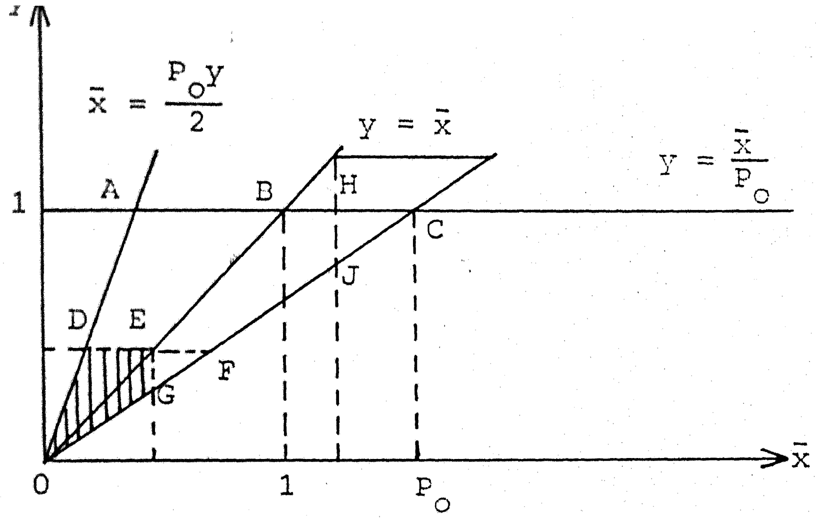

Figure 6.4.1. Showing the region for positive density of $\bar{X}$ and Y for n = 2.

$$= \Pr[\bar{X} \leq z \text{ and } Y \leq z]$$

$$= \Pr[(\bar{X},Y) \in \text{Region ODFO}] - \Pr[(\bar{X},Y) \in \text{Region GEFG}]$$

$$= \frac{4P_o}{P^2}\left[\int_0^z \int_{\frac{P_o y}{2}}^{P_o y} e^{-2\bar{x}}\, d\bar{x}\, dy - \int_{z/P_o}^{z} \int_{z}^{P_o y} e^{-2\bar{x}}\, d\bar{x}\, dy\right]$$

$$= \frac{4P_o}{P^2}\left[\frac{1}{2}\int_0^z (e^{-P_o y} - e^{-2P_o y})\,dy - \frac{1}{2}\int_{z/P_o}^{z} (e^{-2z} - e^{-2P_o y})\,dy\right]$$

$$= \frac{2P_o}{P^2}\left[\frac{1}{2P_o} - \frac{e^{-P_o z}}{P_o} - e^{-2z}\left(z - \frac{z}{P_o} - \frac{1}{2P_o}\right)\right] .$$

(ii) For $1 \leq z \leq P_o$. Now

$$F_Z(z) = 1 - \Pr[(\bar{X},Y) \in \text{Region HJCH}]$$

$$= 1 - \frac{4P_o}{P^2}\int_{z/P_o}^{1} \int_{z}^{P_o y} e^{-2\bar{x}}\, d\bar{x}\, dy$$

$$= 1 - \frac{4P_o}{P^2}\int_{z/P_o}^{1} \frac{1}{2}(e^{-2z} - e^{-2P_o y})\,dy$$

$$= 1 - \frac{2P_o}{P^2}\left[\bar{e}^{2z}\left(1 - \frac{z}{P_o} - \frac{1}{2P_o}\right) + \frac{e^{-2P_o}}{2P_o}\right] .$$

Hence the pdf of Z is given by

$$f_Z(z) = \begin{cases} \dfrac{4P_o}{P^2}\left[\dfrac{e^{-P_o z}}{2} + \bar{e}^{2z}\left(z - \dfrac{z}{P_o} - \dfrac{1}{2}\right)\right] & 0 \leq z \leq 1 \\[2ex] \dfrac{4P_o}{P^2}\,\bar{e}^{2z}\left(1 - \dfrac{z}{P_o}\right) & 1 \leq z \leq P_o . \end{cases}$$

From the pdf of Z given above, we can evaluate E(Z) and $E(Z^2)$.

These are given by

$$E(Z) = \frac{4P_o}{P^2}\left[\frac{1}{8} - \frac{1}{4P_o} + \frac{1}{2P_o^2} - \frac{e^{-P_o}}{2P_o}\left(1 + \frac{1}{P_o}\right) + \frac{e^{-2P_o}}{4}\left(1 + \frac{1}{P_o}\right) - \frac{\bar{e}^2}{8}\right]$$

and

$$E(Z^2) = \frac{4P_o}{P^2}\left[\frac{1}{P_o^3} + \frac{1}{4} - \frac{3}{8P_o} - \frac{e^{-P_o}}{P_o}\left(\frac{1}{2} + \frac{1}{P_o} + \frac{1}{P_o^2}\right) + \right.$$
$$\left. \frac{e^{-2P_o}}{2}\left(\frac{P_o}{2} + 1 + \frac{3}{4P_o}\right) - \frac{\bar{e}^2}{2}\right].$$

From these expressions, the bias and mse of Z can be calculated.

<u>Case 2</u>: n = 3. For this case, we take $1 \leq P_o \leq 3$. Equation (6.4.1) now reduces to

$$f(\bar{x},y) = \begin{cases} \dfrac{9P_o}{P^2}\,\bar{e}^{3\bar{x}}(3\bar{x}-P_o y) = f_1(\bar{x},y) \text{ (say)} & P_o y \leq 3\bar{x} \leq 2P_o y \text{ and } 0 \leq y \leq 1 \\[2ex] \dfrac{9P_o}{P^2}\,\bar{e}^{3\bar{x}}(3P_o y-3\bar{x}) = f_2(\bar{x},y) \text{ (say)} & 2P_o y \leq 3\bar{x} \leq 3P_o y \text{ and } 0 \leq y \leq 1 . \end{cases}$$

It turns out that now two cases depending on the value of P_o

have to be considered separately. These are corresponding to values of $P_o \in [1, 1.5]$, and $P_o \in [1.5, 3]$.

(i) For $1 \le P_o \le 3/2$, the region of positive density of $\bar{X}$ and Y is shown in Figure 6.4.2. In this case for $0 \le z \le 1$,

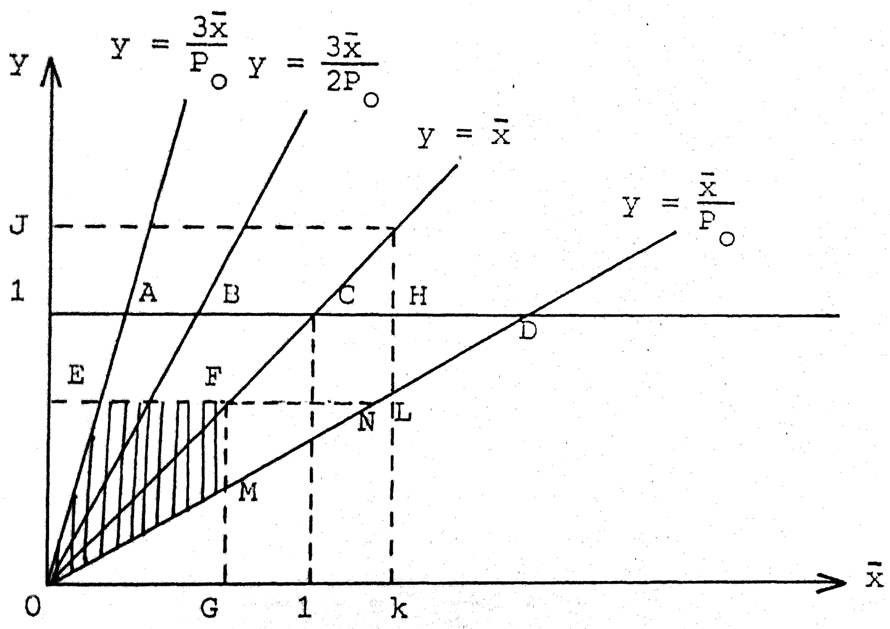

Figure 6.4.2. Showing the region for positive density of $\bar{X}$ and Y for n = 3.

$$F_Z(z) = \Pr[(\bar{X},Y) \in \text{Region EONE}] - \Pr[(\bar{X},Y) \in \text{Region FNMF}]$$

$$= \int_0^z \int_{\frac{P_oy}{3}}^{\frac{2P_oy}{3}} f_1(\bar{x},y)\,d\bar{x}\,dy + \int_0^z \int_{\frac{2P_oy}{3}}^{P_oy} f_2(\bar{x},y)\,d\bar{x}\,dy$$

$$- \int_{z/x_o}^{z} \int_{z}^{P_oy} f_2(\bar{x},y)\,d\bar{x}\,dy$$

$$= \frac{9P_o}{P^3}\Big[\Big(\frac{1}{12P_o} - \frac{e^{-P_oz}}{3P_o} + \frac{e^{-2P_oz}}{4P_o} + \frac{z\,e^{-2P_oz}}{6}\Big) + \Big(\frac{1}{36P_o} - \frac{e^{-3P_oz}}{9P_o} +$$

$$\frac{e^{-2P_o z}}{12P_o} - \frac{z}{6} e^{-2P_o z}) - \{\frac{P_o z^2 \bar{e}^{3z}}{2} (1- \frac{1}{P_o^2}) - (z + \frac{1}{3})\bar{e}^{3z} z(1- \frac{1}{P_o})$$

$$- \frac{1}{9P_o}(e^{-3P_o z} - \bar{e}^{3z})\}]$$

$$= \frac{9P_o}{P^3} [\frac{1}{9P_o} - \frac{e^{-P_o z}}{3P_o} + \frac{e^{-2P_o z}}{3P_o} + \bar{e}^{3z}\{z^2(1- \frac{1}{2P_o} - \frac{P_o}{2}) +$$

$$z(\frac{1}{3} - \frac{1}{3P_o}) - \frac{1}{9P_o}\}] .$$

Similarly, for $1 \leq z \leq P_o$,

$$F_Z(z) = 1 - \int_{z/P_o}^{1} \int_{z}^{P_o y} f_2(\bar{x},y)d\bar{x}\, dy = 1\text{-}Pr[(\bar{X},Y) \in \text{Region HLDH}]$$

$$= 1 - \frac{9P_o}{P^3} [- \frac{e^{-3P_o}}{9P_o} + \bar{e}^{3z}(\frac{1}{9P_o} + \frac{P_o}{2} - z - \frac{1}{3} + \frac{z^2}{2P_o} + \frac{z}{3P_o})]$$

$$= \frac{9P_o}{P^3} [\frac{P^3}{9P_o} + \frac{e^{-3P_o}}{9P_o} - \bar{e}^{3z}(\frac{1}{9P_o} + \frac{P_o}{2} - z - \frac{1}{3} + \frac{z^2}{2P_o} + \frac{z}{3P_o})].$$

Thus the pdf of Z, E(Z) and E(Z^2) are given by

$$f_Z(z) = \begin{cases} \frac{9P_o}{P^3}[\frac{e^{-P_o z}}{3} - \frac{2e^{-2P_o z}}{3} + \bar{e}^{3z}\{\frac{1}{3} + z(1-P_o)-3z^2(1- \frac{P_o}{2} - \frac{1}{2P_o})\}] & 0 \leq z \leq 1 \\ \frac{9P_o}{P^3} \bar{e}^{3z} (\frac{3}{2}P_o - 3z + \frac{3}{2P_o} z^2) & 1 \leq z \leq P_o , \end{cases}$$

$$E(Z) = \frac{9P_o}{P^3}[\frac{P^3}{9P_o} + \frac{1}{6P_o^2} - \frac{1}{9} + \frac{P_o}{27} - \frac{e^{-P_o}}{3P_o^2} + \frac{e^{-2P_o}}{6P_o^2} - \frac{e^{-3P_o}}{9} +$$

$$\bar{e}^3(\frac{2}{9} - \frac{4P_o}{27})] ,$$

$$E(Z^2) = \frac{9P_o}{P^3}\Big[\frac{1}{2P_o^3} + \frac{2P_o}{27} + \frac{4}{27P_o} - \frac{16}{81} - e^{-P_o}\Big(\frac{1}{3P_o} + \frac{2}{3P_o^2} + \frac{2}{3P_o^3}\Big) +$$

$$\frac{e^{-2P_o}}{3P_o}\Big(1 + \frac{1}{P_o} + \frac{1}{2P_o^2}\Big) - \frac{e^{-3P_o}}{9}\Big(P_o + 2 + \frac{4}{3P_o}\Big) + e^{-3}\Big(\frac{64}{81} - \frac{28}{54}P_o\Big)\Big].$$

(ii) For $\frac{3}{2} \le P_o \le 3$, the region of positive density of $\bar{X}$ and Y is shown in Figure 6.4.3. Now for $0 \le z \le 1$,

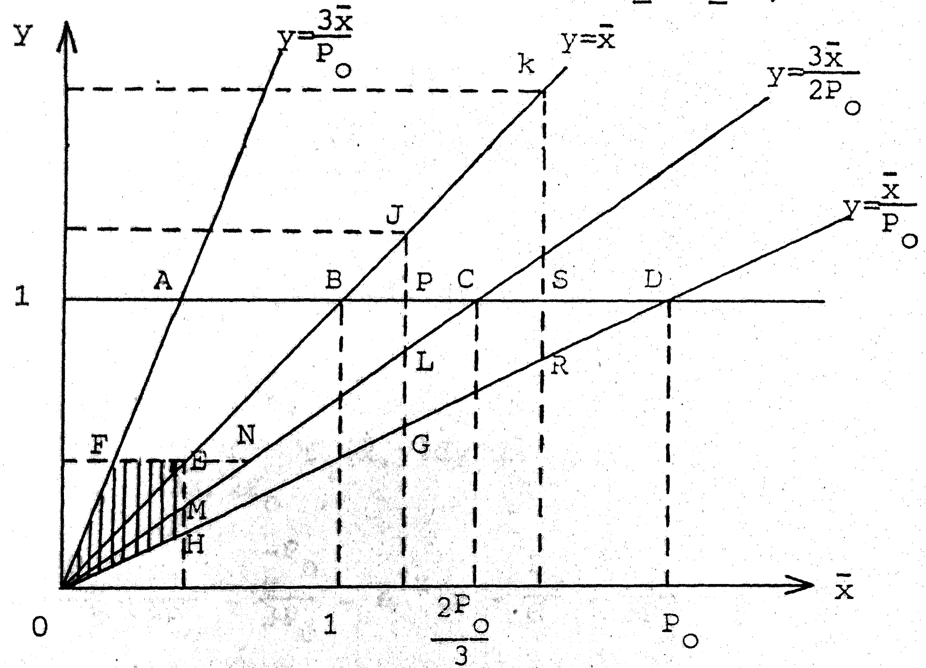

Figure 6.4.3. Showing the region of a positive density of $\bar{X}$ and Y for $n = 3$.

$F_Z(z) = Pr[(\bar{X},Y) \in \text{Region OFNO}] - Pr[(\bar{X},Y) \in \text{Region NEMN}] +$

$Pr[(\bar{X},Y) \in \text{Region OMHO}]$

$$= \int_0^z \int_{\frac{P_o y}{3}}^{\frac{2P_o y}{3}} f_1(\bar{x},y)\,d\bar{x}\,dy + \int_0^z \int_{\bar{x}/P_o}^{3\bar{x}/2P_o} f_2(\bar{x},y)\,dy\,d\bar{x} -$$

$$\int_{3z/2P_o}^{z} \int_{z}^{2P_o y/3} f_1(\bar{x},y)\,d\bar{x}\,dy$$

$$= \frac{9P_o}{p^3}\left[\left(\frac{1}{12P_o} - \frac{e^{-P_o z}}{3P_o} + \frac{e^{-2P_o z}}{4P_o} + \frac{z}{6}e^{-2P_o z}\right) + \left\{\frac{1}{36P_o} - \right.\right.$$

$$\left.\frac{\bar{e}^{3z}}{8P_o}\left(z^2 + \frac{2z}{3} + \frac{2}{9}\right)\right\} - \left\{z\bar{e}^{3z}\left(z + \frac{1}{3} - \frac{3z}{2P_o} - \frac{1}{2P_o}\right) + \frac{e^{-2P_o z}}{6P_o} - \right.$$

$$\frac{\bar{e}^{3z}}{6P_o} - \frac{P_o z^2}{6}e^{-3z}\left(1 - \frac{9}{4P_o^2}\right) + \frac{z}{6}e^{-2P_o z} - \frac{z}{4P_o}\bar{e}^{3z} +$$

$$\left.\left.\frac{1}{12P_o}e^{-2P_o z} - \frac{\bar{e}^{3z}}{12P_o}\right\}\right].$$

Similarly for $1 \leq z \leq \frac{2P_o}{3}$,

$$F_Z(z) = \int_0^1 \int_{P_o y/3}^{2P_o y/3} f_1(\bar{x},y)\,d\bar{x}\,dy + \int_0^z \int_{\bar{x}/P_o}^{3\bar{x}/2P_o} f_2(\bar{x},y)\,dy\,d\bar{x} -$$

$$\int_z^{2x_o/3} \int_{3\bar{x}/2P_o}^{1} f_1(\bar{x},y)\,dy\,d\bar{x}$$

$$= \frac{9P_o}{p^3}\left[\frac{1}{9P_o} - \frac{e^{-P_o}}{3P_o} - \bar{e}^{3z}\left(z - \frac{P_o}{6} - \frac{2z}{3P_o} + \frac{1}{3} - \frac{z^2}{P_o} - \frac{2}{9P_o}\right)\right]$$

and for $\frac{2P_o}{3} \leq z \leq P_o$,

$$F_Z(z) = 1 - \int_{z/x_o}^{1} \int_z^{P_o y} f_2(\bar{x},y)\,d\bar{x}\,dy = 1 - \Pr[(\bar{X},Y) \in \text{Region RSDR}]$$

$$= 1 - \left[-\frac{e^{-3P_o}}{9P_o} + \frac{9P_o}{p^3}\bar{e}^{3z}\left(\frac{P_o}{2} - \frac{z^2}{2P_o} + \frac{1}{9P_o} - \frac{1}{3} + \frac{z}{3P_o} - z + \frac{z^2}{P_o}\right)\right]$$

$$= 1 + \frac{e^{-3P_o}}{p^3} - \frac{9P_o}{p^3}\bar{e}^{3z}\left(\frac{P_o}{2} + \frac{z^2}{2P_o} + \frac{1}{9P_o} - \frac{1}{3} + \frac{z}{3P_o} - z\right).$$

This gives $f_Z(z)$, $E(Z)$ and $E(Z^2)$ as

$$f_Z(z) = \begin{cases} \dfrac{9P_o}{P^3}\left[\dfrac{e^{-P_o z}}{3} + \bar{e}^{3z}\{-\dfrac{1}{3} + z(\dfrac{P_o}{3} - 1) - z^2(\dfrac{3}{P_o} + \dfrac{P_o}{2} - 3)\}\right] & 0 \leq z \leq 1 \\ \dfrac{9P_o}{P^3}\,\bar{e}^{3z}(3z - \dfrac{P_o}{2} - \dfrac{3z^2}{P_o}) & 1 \leq z \leq \dfrac{2P_o}{3} \\ \dfrac{9P_o}{P^3}\,\bar{e}^{3z}(\dfrac{3P_o}{2} + \dfrac{3z^2}{2P_o} - 3z) & \dfrac{2P_o}{3} \leq z \leq P_o , \end{cases}$$

$$E(Z) = \frac{9P_o}{P^3}\left[\frac{1}{9} + \frac{3}{9P_o^2} - \frac{2}{9P_o} - \frac{P_o}{81} - \frac{e^{-P_o}}{3P_o}(1 + \frac{1}{P_o}) + e^{-2P_o}(\frac{1}{3P_o} + \frac{2}{9}) - \frac{e^{-3P_o}}{9}(1 + \frac{1}{P_o}) + \bar{e}^{3}(\frac{4P_o}{81} - \frac{2}{9})\right] ,$$

$$E(Z^2) = \frac{9P_o}{P^3}\left[(\frac{2}{3P_o^3} + \frac{16}{81} - \frac{2P_o}{81} - \frac{8}{27P_o}) - \frac{e^{-P_o}}{3P_o}(1 + \frac{2}{P_o} + \frac{2}{P_o^2}) + \frac{e^{-2P_o}}{27}(4P_o + 12 + \frac{12}{P_o}) - \frac{e^{-3P_o}}{9}(P_o + 2 + \frac{4}{3P_o}) + \bar{e}^{3}(\frac{14}{81}P_o - \frac{64}{81})\right] .$$

In this manner, the distribution of Z can be obtained for higher values of n as well. But the algebra involved becomes lengthy.

6.5 Optimum coefficients for linear estimator $T = a\bar{X} + bX_{(n)}$

For known P, the cdf of $Y_i = \dfrac{X_i}{P_o}$ is

$$F_{Y_i}(y;\sigma) = \begin{cases} 0 & y < 0 \\ \frac{1}{P}[1 - e^{-P_o y/\sigma}] & 0 \leq y \leq \sigma \\ 1 & y > \sigma . \end{cases}$$

taken to be as $Y_{(n)} = \frac{X_{(n)}}{p_o}$ (Robson and Whitlock, 1964; David, 1981, p. 127). However, any estimator based on the largest observation alone is not likely to perform well if there are some outliers in the data.

As seen earlier in Section 6.3, $(\bar{X}, X_{(n)})$ is sufficient for σ and the mle is $Z = \max(\bar{X}, \frac{X_{(n)}}{p_o})$. We therefore look for some alternative estimators of σ based on $\bar{X}$ and $X_{(n)}$, such as $a\bar{X} + bX_{(n)}$. In this section, we first evaluate the coefficients a and b for the linear estimator $T = a\bar{X} + bX_{(n)}$, $a > 0$, $b > 0$, of σ such that the mse (T) is minimum. Again wlog, we take $\sigma = 1$. Then

$$\begin{aligned} \text{mse}(T) &= E(T-1)^2 \\ &= E(a\bar{X} + bX_{(n)} - 1)^2 \\ &= a^2E(\bar{X}^2) + b^2E(X_{(n)}^2) + 1 + 2abE(\bar{X}\,X_{(n)}) - \\ &\quad 2aE(\bar{X}) - 2bE(X_{(n)}). \end{aligned} \tag{6.5.1}$$

Differentiating it with respect to a and b and equating to zero, we get the equations

$$aE(\bar{X}^2) + bE(\bar{X}\,X_{(n)}) = E(\bar{X}) = \nu, \tag{6.5.2}$$

and

$$aE(\bar{X}\,X_{(n)}) + bE(X_{(n)}^2) = E(X_{(n)}) = \nu_{n:n}, \tag{6.5.3}$$

where $\nu = E(X)$ and $\nu_{n:n} = E(X_{(n)})$.

Solving equations (6.5.2) and (6.5.3), we have

$$b = \frac{\nu_{n:n}\,E(\bar{X}^2) - \nu E(\bar{X}\,X_{(n)})}{E(\bar{X}^2)\,E(X_{(n)}^2) - (E(\bar{X}\,X_{(n)}))^2} \tag{6.5.4}$$

$$a = \frac{\nu\, E(X_{(n)}^2) - \nu_{n:n}\, E(\bar{X}\, X_{(n)})}{E(\bar{X}^2)\, E(X_{(n)}^2) - (E(\bar{X}\, X_{(n)}))^2} . \tag{6.5.5}$$

Note that by Cauchy-Schwartz inequality, the denominator term is positive for $n \geq 2$.

Equations (6.5.4) and (6.5.5) give the optimum weights a and b for the linear estimator $T_{opt} = a\bar{X} + bX_{(n)}$. For various values of n and P, these can be easily calculated using the methods described by Saleh et al. (1975). Finally, the mse of T_{opt} can be evaluated by using the formula

$$\begin{aligned} mse(T_{opt}) &= E\, a\bar{X}(a\bar{X} + bX_{(n)} - 1) + E\, bX_{(n)}(a\bar{X} + bX_{(n)} - 1) - \\ &\quad E(a\bar{X} + bX_{(n)} - 1) \\ &= a[aE(\bar{X}^2) + bE(\bar{X}\, X_{(n)}] - \nu) + b[aE(\bar{X}\, X_{(n)}) + \\ &\quad bE(X_{(n)}^2) - \nu_{n:n}] - E(a\bar{X} + bX_{(n)}) + 1 . \end{aligned}$$

Using equations (6.5.2) and (6.5.3), it reduces to

$$\begin{aligned} mse(T_{opt}) &= 1 - E(a\bar{X} + bX_{(n)}) \\ &= 1 - a\nu - b\nu_{n:n} \end{aligned}$$

Hence

$$\begin{aligned} 1 - mse(T_{opt}) &= a\nu + b\,\nu_{n:n} \\ &= \frac{E(X_{(n)}^2)\nu^2 + E(\bar{X}^2)\nu_{n:n}^2 - 2\nu_{n:n}\, \nu E(\bar{X}\, X_{(n)})}{E(\bar{X}^2)\, E(X_{(n)}^2) - [E(\bar{X}\, X_{(n)})]^2} , \end{aligned}$$

on using equations (6.5.4) and (6.5.5). It may be noted that these two equations give

$$\frac{a}{b} = \frac{\nu\, E(X_{(n)}^2) - \nu_{n:n}\, E(\bar{X}\, X_{(n)})}{\nu_{n:n}\, E(\bar{X}^2) - \nu\, E(\bar{X}\, X_{(n)})} . \tag{6.5.6}$$

Next we evaluate a_1 and b_1 such that $T_1 = a_1\overline{X} + b_1X_{(n)}$, $a_1 > 0$, $b_1 > 0$, has a minimum variance in the class of all linear estimators of $\overline{X}$ and $X_{(n)}$, which are unbiased for σ, i.e.,

$$E(a_1\overline{X} + b_1X_{(n)}) = 1 \tag{6.5.7}$$

and

$$V(a_1\overline{X} + b_1X_{(n)}) = E(a_1\overline{X} + b_1X_{(n)})^2 - [E(a_1\overline{X} + b_1X_{(n)})]^2 \tag{6.5.8}$$

is minimum. Eliminating a_1 from equation (6.5.8) with the help of equation (6.5.7), we get

$$Q = V(a_1\overline{X} + b_1X_{(n)}) = E\left(\frac{1-b_1\nu_{n:n}}{\nu}\overline{X} + b_1X_{(n)}\right)^2 - 1$$

$$= E\left[\frac{\overline{X}}{\nu} + b_1\left(\frac{\nu X_{(n)} - \overline{X}\,\nu_{n:n}}{\nu}\right)\right]^2 - 1 .$$

This gives the minimizing equation for b_1 as

$\frac{\partial}{\partial b_1} Q = 0$, that is

$$E\left[\frac{\overline{X}\,X_{(n)}}{\nu} - \frac{\nu_{n:n}\overline{X}^2}{\nu^2} + b_1\left(X_{(n)} - \frac{\nu_{n:n}}{\nu}\overline{X}\right)^2\right] = 0 ,$$

which gives

$$b_1 = \frac{\nu_{n:n}\,E(\overline{X}^2) - \nu\,E(\overline{X}\,X_{(n)})}{E(\nu\,X_{(n)} - \nu_{n:n}\,\overline{X})^2} \tag{6.5.9}$$

and

$$a_1 = \frac{1}{\nu}\,(1-\nu_{n:n}b_1) = \frac{\nu\,E(X^2_{(n)}) - \nu_{n:n}\,E(\overline{X}\,X_{(n)})}{E(\nu\,X_{(n)} - \nu_{n:n}\,\overline{X})^2} . \tag{6.5.10}$$

Since $\left.\frac{\partial^2 Q}{\partial b_1^2}\right|_{(a_1,b_1)} = E\left(X_{(n)} - \frac{\nu_{n:n}\overline{X}}{\nu}\right)^2$

> 0 ,

this implies that at a_1 and b_1, variance is minimum. Further $\frac{a}{b} = \frac{a_1}{b_1}$ and

$$\frac{a}{a_1} = \frac{b}{b_1} = 1 - mse(T_{opt}) = a\nu + b\nu_{n:n} , \qquad (6.5.11)$$

and the minimum variance of T_1 is

$$V(T_{1opt}) = E(a_1\bar{X} + b_1X_{(n)} - 1)^2 .$$

On using equation (6.5.11), we have

$$V(T_{1opt}) = E(\frac{a\bar{X}}{1-mse(T_{opt})} + \frac{bX_{(n)}}{1-mse(T_{opt})} - 1)^2$$

$$= \frac{mse(T_{opt})}{1 - mse(T_{opt})} ,$$

on simplification. Thus, the efficiency of T_{opt} relative to T_{1opt} is

$$Eff(T_{opt}, T_{1opt}) = \frac{Var(T_{1opt})}{mse(T_{opt})} = \frac{1}{(1-mse(T_{opt}))} .$$

Hence T_{opt} is more efficient than T_{1opt} by a factor $\frac{1}{1-mse(T_{opt})}$. Further, the coefficients a_1, b_1 can be easily calculated from a and b.

For calculating a and b, we first obtain ν, $E(\bar{X}^2)$, $\nu_{n:n}$, $E(X^2_{(n)})$, $E(X_{(n-1)}\, X_{(n)})$ and $E(\bar{X}\, X_{(n)})$ by using the following formulas given by Saleh et al. (1975).

$$\nu_{n:n} = E(X_{(n)}) = G_o - G_n , \qquad (6.5.12)$$

$$\nu_{n,n:n} = E(X^2_{(n)}) = - G_o\, \nu_{n:n} - [2G_n \sum_{i=1}^{n} \frac{1}{i} - \sum_{i=1}^{n} \frac{G_{n-i}}{i}],$$

$$\nu_{n-1,n:n} = E(X_{(n-1)}X_{(n)}) = G_o^2 - nG_{n-1}(G_o+1) + \sum_{i=1}^{n} \frac{G_{n-i}}{i} +$$

$$G_n[(n-1)G_o + 1 + n - 2 \sum_{i=1}^{n} \frac{1}{i}],$$

where $G_o \equiv P_o = -\log(1-P)$

$$G_n = \frac{1}{P^n}[G_o - \sum_{v=1}^{n} \frac{P^v}{v}] \qquad \text{for } n > 0 .$$

For $n = 1$, we immediately get

$$E(X) = \nu_{1:1} = \nu = 1 - \frac{(1-P)}{P} G_o$$

and

$$E(X^2) = \nu_{1,1:1} = 2 - \frac{(1-P)}{P} G_o[2 - \log(1-P)] . \tag{6.5.13}$$

This gives $E(\overline{X}) = \nu$ and

$$E(\overline{X}^2) = \frac{1}{n} E(X^2) + \nu^2(1 - \frac{1}{n}) .$$

Note that $E(\overline{X}\, X_{(n)}) = \frac{1}{n}E(\sum_{i=1}^{n} X_{(1)}\, X_{(n)}) = \frac{1}{n} \sum_{i=1}^{n} \nu_{i,n:n}$. This can be calculated either from the moment expressions given by Saleh et al. (1975), or simply by using the identity

$$\nu_{n-1,n:n} + \sum_{i=1}^{n-1} \nu_{i,n:n} = n\, \nu_{n-1:n-1}\, \nu_{1:1} ,$$

proved by Joshi and Balakrishnan (1982). This gives

$$E(\overline{X}\, X_{(n)}) = \nu_{n-1:n-1}\, \nu_{1:1} - \frac{1}{n} \nu_{n-1,n:n} + \frac{1}{n} \nu_{n,n:n} .$$

For calculation purposes $\nu_{n:n}$, $\nu_{n,n:n}$ and $\nu_{n-1,n:n}$ are first evaluated for $n = 1,2,3,\ldots$ in this order and other quantities needed for a and b are then evaluated. Some calculations for $P = .8$ and $n = 2(1)10$ are given in Table 6.5.1.

TABLE 6.5.1: The values of a, b, $mse(T_{opt})$ and efficiency of T_{opt} compared to T_{1opt}

n	a	b	$mse(T_{opt})$	Efficiency of T_{opt}
2	.24438	.78628	.18978	1.23424
3	.25410	.73441	.12339	1.14076
4	.25293	.70357	.08815	1.09667
5	.24758	.68333	.06680	1.07158
6	.24058	.66918	.05272	1.05565
7	.23302	.65884	.04286	1.04478
8	.22540	.65102	.03565	1.03697
9	.21796	.64495	.03019	1.03113
10	.21080	.64015	.02595	1.02664

6.6 Exact distribution of $T = a\bar{X} + bX_{(n)}$ for small values of n

The distribution of T is quite complicated. We have to start with the joint pdf of $(\bar{X}, X_{(n)})$ given at equation (6.2.6). Here we evaluate the distribution of T for n = 2 and 3 only.

Case 1: For n = 2, the density of $\bar{X}$ and $X_{(n)}$ given at equation (6.2.6) reduces to

$$f(\bar{x}, x_{(n)}) = \begin{cases} \frac{4}{P^2} e^{-2\bar{x}} & x_{(n)} \le 2\bar{x} \le 2x_{(n)} \text{ and } 0 < x_{(n)} \le P_0 \\ 0 & \text{otherwise.} \end{cases}$$

Make a transformation $W = X_{(n)}$ and $T = a\bar{X} + bX_{(n)}$, a, b > 0, then the Jacobian is $\frac{1}{a}$ and the joint density of T and W is

$$f(t,w) = \begin{cases} \dfrac{4}{P^2 a}\, e^{-2\left(\frac{t-bw}{a}\right)} & \dfrac{aw}{2} + bw \le t \le (b+a)w \\ & \text{and } 0 \le w \le P_o \\ 0 & \text{otherwise .} \end{cases}$$

The region of positive density of T and W is shown in Figure 6.6.1.

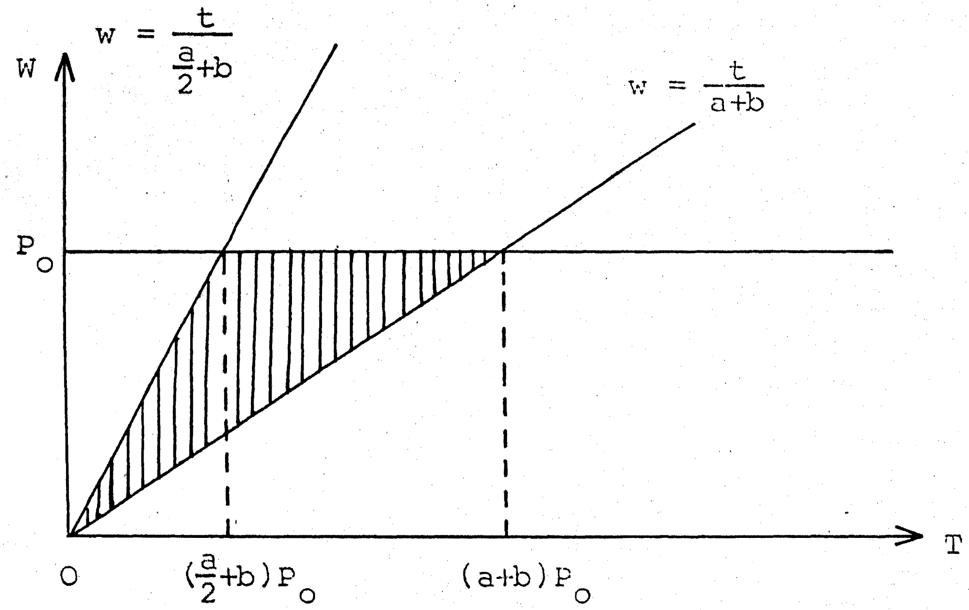

Figure 6.6.1. Showing the region of positive density of T and W for n = 2.

On integrating out w, we get the density of T as follows:

For $0 \le t \le (\frac{a}{2} + b)P_o$,

$$f_T(t) = \int_{t/(a+b)}^{t/(\frac{a}{2}+b)} \frac{4}{aP^2}\, e^{-2\left(\frac{t-bw}{a}\right)}\, dw$$

$$= \frac{4e^{-2t/a}}{aP^2}\left[\frac{a}{2b}\, e^{2bw/a}\right]_{t/(a+b)}^{t/(\frac{a}{2}+b)}$$

$$= \frac{2e^{-2t/a}}{bP^2}\left[e^{\frac{4bt}{a(a+2b)}} - e^{\frac{2bt}{a(a+b)}}\right],$$

and for $(\frac{a}{2} + b)P_o \le t \le (a+b)P_o$,

$$f_T(t) = \int_{t/(a+b)}^{P_o} \frac{4}{aP^2} e^{-2(\frac{t-bw}{a})} dw$$

$$= \frac{4e^{-2t/a}}{aP^2}\left[\frac{a}{2b} e^{\frac{2bw}{a}}\right]_{t/(a+b)}^{P_o}$$

$$= \frac{2e^{-2t/a}}{bP^2}\left[e^{2bP_o/a} - e^{\frac{2bt}{a(a+b)}}\right].$$

Combining, we get

$$f_T(t) = \begin{cases} \dfrac{2e^{-\frac{2t}{a}}}{bP^2}\left[e^{\frac{4bt}{a(a+2b)}} - e^{\frac{2bt}{a(a+b)}}\right] & 0 \le t \le (\frac{a}{2} + b)P_o \\ \dfrac{2e^{-\frac{2t}{a}}}{bP^2}\left[e^{\frac{2bP_o}{a}} - e^{\frac{2bt}{a(a+b)}}\right] & (\frac{a}{2} + b)P_o \le t \le (a+b)P_o . \end{cases}$$

<u>Case 2</u>: $n = 3$. From equation (6.2.6) we have the density of $\bar{X}$ and $X_{(n)}$ as

$$f(\bar{x}, x_{(n)}) = \begin{cases} \dfrac{9e^{-3\bar{x}}(3\bar{x} - x_{(n)})}{P^3} & x_{(n)} \le 3\bar{x} \le 2x_{(n)} \\ & 0 \le x_{(n)} \le P_o \\ \dfrac{9e^{-3\bar{x}}[3(x_{(n)} - \bar{x})]}{P^3} & 2x_{(n)} \le 3\bar{x} \le 3x_{(n)} \\ & 0 \le x_{(n)} \le P_o . \end{cases}$$

Make the transformation $W = X_{(n)}$ and $T = a\bar{X} + bX_{(n)}$ where $a, b > 0$. Then $|J| = \frac{1}{a}$ and we get the joint density of T and W as

$$
f(t,w) = \begin{cases} \dfrac{9e^{-3(\frac{t-bw}{a})}}{aP^3}\left[3\left(\dfrac{t-bw}{a}\right)-w\right] & \left(\dfrac{a}{3}+b\right)w \le t \le \left(\dfrac{2a}{3}+b\right)w \\ & 0 \le w \le P_o \\ \dfrac{9e^{-3(\frac{t-bw}{a})}}{aP^3}\, 3\left[w-\left(\dfrac{t-bw}{a}\right)\right] & \left(\dfrac{2a}{3}+b\right)w \le t \le (a+b)w \\ & 0 \le w \le P_o \end{cases}
$$

$$
= \begin{cases} \dfrac{9e^{-3(\frac{t-bw}{a})}}{aP^3}\left[\dfrac{3t}{a} - w\left(\dfrac{3b+a}{a}\right)\right] & \left(\dfrac{a+3b}{3}\right)w \le t \le \left(\dfrac{2a+3b}{3}\right)w \\ & 0 \le w \le P_o \\ \dfrac{9e^{-3(\frac{t-bw}{a})}}{aP^3}\, 3\left[w\left(\dfrac{a+b}{a}\right) - \dfrac{t}{a}\right] & \left(\dfrac{2a+3b}{3}\right)w \le t \le (a+b)w \\ & 0 \le w \le P_o\,. \end{cases}
$$

The region for the positive density of T and W is shown in Figure 6.6.2.

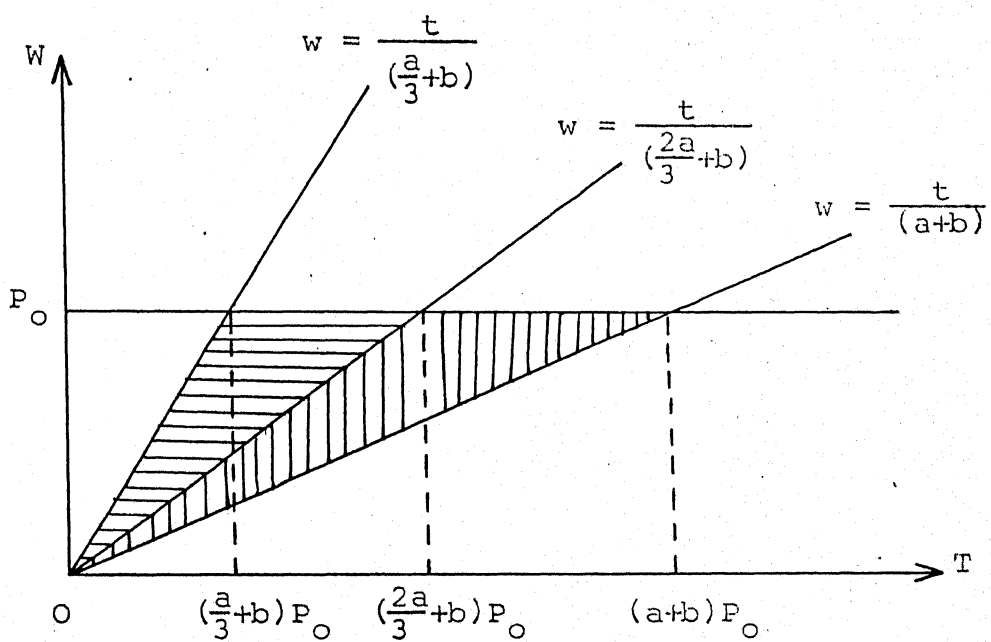

Figure 6.6.2. Showing the region for positive density of T and W for n = 3.

Now we evaluate the density of T by integrating out W as follows.

For $0 \leq t \leq (\frac{a+3b}{3})P_o$,

$$f_T(t) = \frac{9e^{-3t/a}}{aP^3} \int_{\frac{3t}{(2a+3b)}}^{\frac{3t}{(a+3b)}} e^{\frac{3bw}{a}} \left[\frac{3t}{a} - w\left(\frac{3b+a}{a}\right)\right] dw +$$

$$\frac{27e^{-3t/a}}{aP^3} \int_{\frac{t}{(a+b)}}^{\frac{3t}{(2a+3b)}} \left[w\left(\frac{a+b}{a}\right) - \frac{t}{a}\right] e^{\frac{3bw}{a}} dw$$

$$= \frac{9e^{-3t/a}}{aP^3} \left[\left(e^{\frac{9bt}{a(a+3b)}} - e^{\frac{9bt}{a(2a+3b)}}\right)\frac{t}{b} - \left(\frac{3b+a}{a}\right) \frac{a}{3b}\right.$$

$$\cdot \left(\frac{3t}{a+3b} e^{\frac{9bt}{a(a+3b)}} - \frac{a}{3b} e^{\frac{9bt}{a(a+3b)}} - \frac{3t}{(2a+3b)} e^{\frac{9bt}{a(2a+3b)}} +\right.$$

$$\left.\frac{a}{3b}e^{\frac{9bt}{a(2a+3b)}}\right) + 3\left\{\left(\frac{a+b}{a}\right) \frac{a}{3b}\left(\frac{3t}{(2a+3b)} e^{\frac{9bt}{a(2a+3b)}} -\right.\right.$$

$$\left.\frac{a}{3b}e^{\frac{9bt}{a(2a+3b)}} - \frac{t}{(a+b)} e^{\frac{3bt}{(a+b)a}} + \frac{a}{3b} e^{\frac{3bt}{a(a+b)}}\right) -$$

$$\left.\left.\frac{t}{3b} \left(e^{\frac{9bt}{(2a+3b)a}} - e^{\frac{3bt}{a(a+b)}}\right)\right\}\right]$$

$$= \frac{9e^{-3t/a}}{aP^3} \left[e^{\frac{9bt}{a(a+3b)}} a\left(\frac{3b+a}{9b^2}\right) - e^{\frac{9bt}{a(2a+3b)}} \frac{2a(2a+3b)}{9b^2} +\right.$$

$$\left.e^{\frac{3bt}{a(a+b)}} \frac{a(a+b)}{3b^2}\right]$$

$$= \left[\frac{a(a+3b)}{9b^2} e^{-\frac{3t}{a+3b}} - \frac{2a(2a+3b)}{9b^2} e^{-\frac{6t}{2a+3b}} + \frac{a(a+b)}{3b^2} e^{-\frac{3t}{a+b}}\right] \frac{9}{aP^3} .$$

Next for $(\frac{a+3b}{3})P_o \leq t \leq (\frac{2a+3b}{3})P_o$,

$$f_T(t) = \frac{9e^{-\frac{3t}{a}}}{aP^3}\left[\int_{\frac{3t}{(2a+3b)}}^{P_o} e^{\frac{3bw}{a}} \left\{\frac{3t}{a} - w\left(\frac{3b+a}{a}\right)\right\} dw \; + \right.$$

$$\left. 3\int_{\frac{t}{a+b}}^{\frac{3t}{2a+3b}} \left\{w\left(\frac{a+b}{a}\right) - \frac{t}{a}\right\} e^{\frac{3bw}{a}} dw\right]$$

$$= \frac{9e^{-\frac{3t}{a}}}{aP^3}\left[\frac{t}{b}\left(e^{\frac{3bP_o}{a}} - e^{\frac{9bt}{(2a+3b)a}}\right) - \frac{3b+a}{3b}\left(P_o e^{\frac{3bP_o}{a}} - \frac{ae^{\frac{3bP_o}{a}}}{3b}\right.\right.$$

$$\left. - \frac{3t}{(2a+3b)} e^{\frac{9bt}{a(2a+3b)}} + \frac{a}{3b} e^{\frac{9bt}{a(2a+3b)}}\right) + 3\left\{\frac{a+b}{3b} \; X\right.$$

$$\left(\frac{3t}{2a+3b} e^{\frac{9bt}{a(2a+3b)}} - \frac{a}{3b} e^{\frac{9bt}{a(2a+3b)}} - \frac{t}{a+b} e^{\frac{3bt}{(a+b)a}} + \right.$$

$$\left.\left.\left. \frac{a}{3b} e^{\frac{3bt}{(a+b)a}}\right) - \frac{t}{3b}\left(e^{\frac{9bt}{a(2a+3b)}} - e^{\frac{3bt}{(a+b)a}}\right)\right\}\right]$$

$$= \frac{9e^{-3t/a}}{aP^3}\left[e^{\frac{3bP_o}{a}}\left\{\frac{t}{b} - \left(\frac{3b+a}{3b}\right)\left(P_o - \frac{a}{3b}\right)\right\} - e^{\frac{9bt}{a(2a+3b)}} \frac{2a(2a+3b)}{9b^2}\right.$$

$$\left. + e^{\frac{3bt}{a(a+b)}} \frac{a(a+b)}{3b^2}\right]$$

$$= \frac{9}{aP^3}\left[e^{\frac{3bP_o}{a}}\left\{\frac{t}{b} - \left(\frac{3b+a}{3b}\right)\left(P_o - \frac{a}{3b}\right)\right\}e^{-\frac{3t}{a}} - \frac{2a(2a+3b)}{9b^2} e^{-\frac{6t}{2a+3b}}\right.$$

$$\left. + \frac{a(a+b)}{3b^2} e^{-\frac{3t}{a+b}}\right].$$

Similarly for $\left(\frac{2a+3b}{3}\right)P_o \leq t \leq (a+b)P_o$,

$$f_T(t) = \frac{27e^{-3t/a}}{aP^3}\int_{\frac{t}{(a+b)}}^{P_o} e^{\frac{3bw}{a}}\left[\frac{w(a+b)}{a} - \frac{t}{a}\right] dw$$

$$= \frac{27\bar{e}^{3t/a}}{aP^3}\left[\frac{a+b}{3b}\left(P_o e^{\frac{3bP_o}{a}} - \frac{a}{3b} e^{\frac{3bP_o}{a}} - \frac{t}{(a+b)} e^{\frac{3bt}{(a+b)a}} + \frac{a}{3b} e^{\frac{3bt}{a(a+b)}}\right) - \frac{t}{3b}\left(e^{\frac{3bP_o}{a}} - e^{\frac{3bt}{a(a+b)}}\right)\right]$$

$$= \frac{9\bar{e}^{3t/a}}{aP^3}\left[e^{\frac{3bP_o}{a}}\left\{\frac{(a+b)}{b} P_o - \frac{(a+b)a}{3b^2} - \frac{t}{b}\right\} + e^{\frac{3bt}{a(a+b)}} \frac{a}{3b} \frac{a+b}{b}\right]$$

$$= \frac{9}{aP^3}\left[e^{\frac{3bP_o}{a}}\left\{\left(\frac{a+b}{b}\right)P_o - \frac{(a+b)a}{3b^2} - \frac{t}{b}\right\}e^{-3t/a} + \frac{a(a+b)}{3b^2} \bar{e}^{\frac{3t}{a+b}}\right].$$

The distribution of T for higher values of n can be obtained in a similar manner.

6.7 Other estimators and comparison of various estimators in a single outlier case

In Section 6.5, we have obtained an estimator $T_{opt} = a\bar{X} + bX_{(n)}$ which has the minimum mse among all linear estimators of $\bar{X}$ and $X_{(n)}$, where a and b are given by equations (6.5.4) and (6.5.5).

Let there be an outlier in the sample, i.e., in an independent sample of size n, (n-1) observations are from the distribution with pdf $f(x;\sigma)$ given at equation (6.1.1) and one comes from the distribution with pdf $g(x:\sigma) \equiv f(x; \frac{\sigma}{\alpha})$, $\alpha > 0$. This makes the distribution theory of order statistics as extremely complicated due to different ranges for (n-1) observations and the outlying observation. In actual situations, this case is not likely to occur as often as the case of outliers with known truncation point x_o.

In Chapter 4, we have seen that U_4 performs better for small values of α. Since for small α, $X_{(n)}$ has the largest

probability of being an outlier; hence $X_{(1)},\ldots,X_{(n-1)}$ may be considered as order statistics from $f(x;\sigma)$ in this truncated case with

$$E(U_4) = \frac{1}{n-1}\sum_{i=1}^{n-1} E(X_{(i)}) \doteq P_1\sigma , \qquad (6.7.1)$$

where $P_1 = 1 + (\frac{1-P}{P}) \log(1-P)$ on using equation (6.5.13). Intuitively it suggests that U_4/P_1 can also be used for estimating σ. The estimators which are included in the present study are T_{opt}, U_4/P_1, mle discussed in Section 6.3 and $T_1 = a_{n-1}\bar{X}_{n-1} + b_{n-1}X_{(n-1)}$, where a_{n-1}, b_{n-1} and $\bar{X}_{n-1}$ are obtained from the sample after deleting $X_{(n)}$. The last estimator T_1 can be justified on the grounds that for $\alpha < 1$, $X_{(n)}$ has the largest probability of being an outlier and hence by deleting $X_{(n)}$ and obtaining a estimator of σ similar to T_{opt} which uses only $X_{(1)},\ldots,X_{(n-1)}$, we may get a reasonable estimator. We have obtained the biases and mse's of these estimators by simulation using 1000 iterations for $\sigma = 1$, $n = 10$ and 20, $P = .8$, .9, .95, .99 and $\alpha = .1$, .2, .5, 1 and 2 which are given in the Table 6.7.1 and Table 6.7.2. These values are approximately same as corresponding exact values of those estimators which are available. Table 6.7.2 shows that T_{opt} is best among all estimators considered for $\alpha = 1$ and large α values. For small values of α, T_1 performs fairly well. It is clear from the table that mle does not perform good compared to others. Table 6.7.2 also reveals that if $P > .95$, we may analyze the data by the technique used in untruncated case because as P becomes greater than .95, T_{opt} is better for large α values

and U_4/P_1 performs good for small α values. This is similar to the case for untruncated exponential distribution discussed in Chapter 4 as well because as $P \to 1$, $a\bar{X} + bX_{(n)} \to \sum_{i=1}^{n} X_i/(n+1)$ and $\frac{U_4}{P_1} \to U_4$.

We now restrict our attention to the values of P which are less than or equal to .95, since for larger P we may use the techniques of untruncated case. The above observations suggest an estimator which depends upon α. But parameter α is itself unknown. So we have to find some reasonable estimators of α. If α is small then $x_{(n)}$ has the largest probability of being an outlier and $X_{(n)}$ may be considered as coming from $f(x;\ \sigma/\alpha)$. Consequently from equation (6.5.13), we have

$$E(X_{(n)}) \doteq P_1\ \sigma/\alpha\ ,$$

and $E(U_4)$ given at equation (6.7.1). This gives an estimator of α as $\hat{\alpha} = \frac{E(U_4)}{E(X_{(n)})}$. Finally, we have an estimator of σ as

$$T_2 = \begin{cases} a\bar{X} + bX_{(n)} & \dfrac{U_4}{X_{(n)}} > c \\ a_{n-1}\bar{X}_{n-1} + b_{n-1}X_{(n-1)} & \dfrac{U_4}{X_{(n)}} \le c\ , \end{cases}$$

where

$$c = \left.\frac{E(U_4)}{E(X_{(n)})}\right|_{\alpha=1} = \left.\frac{\frac{1}{n-1}\ E\ \sum_{i=1}^{n-1} X_{(i)}}{E\ X_{(n)}}\right|_{\alpha=1} = \left.\frac{\frac{1}{n-1}\ E[n\bar{X}-X_{(n)}]}{E(X_{(n)})}\right|_{\alpha=1}$$

$$= \frac{1}{n-1}\ \frac{n\nu-\nu_{n:n}}{\nu_{n:n}}$$

and the expressions of $\nu_{n:n}$ and ν are given in equations (6.5.12) and (6.5.13) respectively.

We have obtained the bias and mse of T_2 by simulation using 1000 iterations for the same combination of n, P and α as for other estimators. These values are also given in Tables 6.7.1 and 6.7.2. These simulated values show that from mse point of view T_2 is better than T_1 for $\alpha \geq .2$ and for small α values T_1 is only marginally better than T_2. For large values of α, T_{opt} is better than T_2, while T_2 is better than T_{opt} for small α values. In general, from the bias consideration, U_4/P_1 is best among all estimators studied. However, T_2 is also not far behind. Consequently we recommend the estimator T_2 if one wants to protect against one outlier which has a higher expectation. The estimater T_{opt} is recommended for use only if the experimenter feels that the data does not contain any outliers.

6.8 Estimation of scale parameter of a truncated exponential distribution in a single outlier case when x_o is known

Estimation of scale parameter of the truncated exponential distribution for this case of known truncation point x_o with pdf given at equation (6.1.2) is considered by Deemer and Votaw (1955). They have obtained mle of σ which is as follows:

If $\frac{\bar{x}}{x_o} < \frac{1}{2}$, then mle of σ will be the solution of the likelihood equation

$$\sigma = \bar{x} + \frac{x_o}{(e^{x_o/\sigma} - 1)} \tag{6.8.1}$$

otherwise mle of $\sigma = \infty$. Deemer and Votaw (1955) have also shown that equation (6.8.1) has a unique solution. Note that

a finite estimator of σ is obtained only if $\frac{\bar{x}}{x_o} < \frac{1}{2}$. Therefore we consider some other estimators of σ. We may use the method of moments for estimation of σ. But the method of maximum likelihood and method of moments based on first moment are same. Therefore method of moments based on higher order moments can be used (see, Schneider, 1986, p. 39).

6.8.1 Method of moments

Using equation (2.1.4), we have

$$E(X^k) = \frac{x_o^k}{1-\bar{e}^b}\left[\frac{(k)!}{b^k}(1-\bar{e}^b) - \bar{e}^b\{1 + \sum_{i=1}^{k-1}\frac{k(k-1)\ \ldots\ (k-i+1)}{b^i}\}\right], \tag{6.8.2}$$

where $b = \frac{x_o}{\sigma}$.

In particular

$$E(X) = \sigma - \frac{x_o}{e^{x_o/\sigma} - 1} = \sigma\left[1 - \frac{x_o/\sigma}{e^{x_o/\sigma} - 1}\right]$$

and $$E(X^2) = \sigma^2\left[2 - \frac{(x_o/\sigma + 2)x_o/\sigma}{e^{x_o/\sigma} - 1}\right] . \tag{6.8.3}$$

Moment estimaor of σ based on kth order moment can be obtained by solving equation

$$\frac{1}{nx_o^k}\sum_{i=1}^{n} x_i^k = \frac{(k)!}{b^k} - \frac{\bar{e}^b}{1-\bar{e}^b}\left[1 + \sum_{i=1}^{k-1}\frac{k(k-1)\ \ldots\ (k-i+1)}{b^i}\right], \tag{6.8.4}$$

for b and hence for $\sigma = \frac{x_o}{b}$. R.H.S. of this equation is a function of b only, so we may write it as

$$\frac{1}{nx_o^k}\sum_{i=1}^{n} x_i^k = h(b) ,$$

where $h(b) = \frac{(k)!}{b^k} - \frac{1}{e^b-1}\{1 + \sum_{i=1}^{k-1} \frac{k(k-1)\ \dots\ (k-i+1)}{b^i}\}$.

Clearly

$$\lim_{b\to\infty} h(b) = 0.$$

Next $h(b)$ can be written as

$$h(b) = \frac{e^{-b}}{b^k(1-e^{-b})}[(k)!(e^b-1) - b^k - b^k \sum_{i=1}^{k-1} \frac{k(k-1)\ \dots\ (k-i+1)}{b^i}]$$

$$= \frac{1}{b^k(e^b-1)}[\{(k)!b + \frac{(k)!b^2}{(2)!} + \frac{(k)!}{(3)!}b^3 + \dots + \frac{(k)!}{(k-1)!}b^{k-1} +$$

$$\frac{(k)!}{(k)!}b^k + \frac{(k)!\ b^{k+1}}{(k+1)!} + \dots\} - b^k - \{kb^{k-1} + k(k-1)b^{k-2}$$

$$+ \dots + (k)!\ b^1\}]\ .$$

After simple simplification, we have

$$h(b) = \frac{1}{b^k(e^b-1)}[\frac{b^{k+1}}{(k+1)!} + \frac{b^{k+2}}{(k+2)!} + \dots]$$

$$= \frac{[\frac{(k)!}{(k+1)!} + \frac{b(k)!}{(k+2)!} + \dots]}{[\frac{1}{(1)!} + \frac{b}{(2)!} + \dots]}\ .$$

Therefore

$$\lim_{b\to 0} h(b) = \frac{1}{k+1}\ .$$

Now we want to know whether or not $h(b)$ is an increasing function. To this end, note that

$$h'(b) = -\frac{k(k)!}{b^{k+1}} + \frac{e^b}{(e^b-1)^2}\{1 + \sum_{i=1}^{k-1} \frac{k(k-1)\ \dots\ (k-i+1)}{b^i}\}$$

$$+ \frac{1}{(e^b-1)} \sum_{i=1}^{k-1} \frac{k(k-1)\ \dots\ (k-i+1)i}{b^{i+1}}$$

$$= \frac{1}{b^{k+1}(e^b-1)^2} [-k(k)!(e^b-1)^2 + e^b b^{k+1} \{1 + \sum_{i=1}^{k-1}$$

$$. \frac{k(k-1) \ldots (k-i+1)}{b^i}\} + b^{k+1}(e^b-1)\{\sum_{i=1}^{k-1} \frac{k(k-1) \ldots (k-i+1)i}{b^{i+1}}\}]$$

$$= \frac{1}{b^{k+1}(e^b-1)^2} [(e^b-1)\{b^{k-1}k + 2b^{k-2}k(k-1) + 3k(k-1)(k-2)$$

$$. b^{k-3} + \ldots + (k-1)(k)!b - k(k)!(\frac{b}{(1)!} + \frac{b^2}{(2)!} + \frac{b^3}{(2)!} + \ldots)\}$$

$$+ e^b b^{k+1} + e^b b^{k+1} \sum_{i=1}^{k-1} \frac{k(k-1) \ldots (k-i+1)}{b^i}]$$

$$= \frac{1}{b^{k+1}(e^b-1)^2} [-(e^b-1) \{b(k)! + \frac{2(k)!b^2}{(2)!} + \frac{3(k)!b^3}{(3)!} + \ldots +$$

$$(k-1)\frac{(k)!b^{k-1}}{(k-1)!} + \frac{k(k)!b^k}{(k)!} + \frac{k(k)!b^{k+1}}{(k+1)!} + \ldots\} +$$

$$e^b\{b^{k+1} + kb^k + k(k-1)b^{k-1} + \ldots + \frac{(k)!}{(2)!} b^3 + (k)!b^2\}].$$

On rearranging the terms, we get

$$h'(b) = \frac{1}{b^{k+1}(e^b-1)^2} [b^{k+1} + kb^k + k(k-1)b^{k-1} + \ldots + \frac{(k)!}{(2)!} b^3 +$$

$$(k)!b^2 - (e^b-1)\{b(k)! + \frac{k(k)!}{(k+1)!} b^{k+1} + \frac{k(k)!}{(k+2)!} b^{k+2}$$

$$+ \ldots - b^{k+1}\}]$$

$$= \frac{1}{b^{k+1}(e^b-1)^2} [-\{b(k)!(e^b-1) - b^{k+1} - kb^k - \frac{(k)!}{(k-2)!}b^{k-1} - \ldots -$$

$$(k)!b^2\} - k(k)!(e^b-1)\{\frac{b^{k+1}}{(k+1)!} + \frac{b^{k+2}}{(k+2)!} + \ldots - \frac{b^{k+1}}{k(k)!}].$$

On expanding e^b in terms of b, it reduces to

$$= \frac{-1}{b^{k+1}(e^b-1)^2}\,[(k)!\,b\{\frac{b^{k+1}}{(k+1)!} + \frac{b^{k+2}}{(k+2)!} + \ldots\} +$$

$$k(k)!\ \{-\frac{b^{k+1}}{k(k)!} + \frac{b^{k+1}}{(k+1)!} + \ldots\}\ (e^b-1)]$$

$$= \frac{-1}{b^{k+1}(e^b-1)^2}\,[(k)!\,b(\frac{b^{k+1}}{(k+1)!} + \frac{b^{k+2}}{(k+2)!} + \ldots) + k(k)!$$

$$\cdot(-\frac{b^{k+1}}{k(k+1)!} + \frac{b^{k+2}}{(k+2)!} + \frac{b^{k+3}}{(k+3)!} + \ldots)(\frac{b}{(1)!} + \frac{b^2}{(2)!} +$$

$$\frac{b^3}{(3)!} + \ldots)]$$

$$= \frac{-1}{b^{k+1}(e^b-1)^2}\sum_{i=2}^{\infty} b^{k+1+i}\,[\frac{(k)!}{(k+i)!} + \frac{(k)!}{(k+2)!(i)!}((i-1)k-2)$$

$$+ k(k)!(\frac{1}{(k+3)!(i-2)!} + \ldots + \frac{1}{(2)!(k+i-1)!} + \frac{1}{(1)!(k+i)!})]$$

$$< 0.$$

This implies that h(b) is a monotonically decreasing function of b and we have seen that $\lim_{b\to 0} (b) = \frac{1}{k+1}$ which gives $h(b) < \frac{1}{k+1}$.

It implies that a unique solution of equation (6.8.4) exists if $\frac{\Sigma x_i^k}{n x_o^k} < \frac{1}{k+1}$. Otherwise it has no solution and an estimator of b may be taken as 0 and hence of σ as ∞. This generalizes the result of Deemer and Votaw (1955) for k = 1 when the method of moments and method of maximum likelihood are identical.

Now we give some examples such that the solution of equation (6.8.4) exists for k = 2 but not for k = 1 and vice versa. There are also cases where both the solutions exist or do not exist .

EXAMPLE 1: Suppose $n = 11$, $x_o = 1$ and the sample is such that $x_i = 0$ for $i = 1,\ldots,6$ and $x_i = 1$ for $i = 7,\ldots,11$. In this case $\bar{x} = \frac{5}{11} < \frac{1}{2}$ but $\frac{\Sigma x_i^2}{n} = \frac{5}{11} > \frac{1}{3}$ and hence equation (6.8.4) has a solution for $k = 1$ but not for $k = 2$.

EXAMPLE 2: Suppose $n = 10$, $x_o = 1$ and the sample is such that $x_i = \frac{1}{2}$ for all $i = 1,..,10$. In this case $\bar{x} = \frac{1}{2} \geq \frac{1}{2}$ but $\frac{\Sigma x_i^2}{n} = \frac{1}{4} < \frac{1}{3}$ and hence equation (6.8.4) has a solution for $k = 2$ but not for $k = 1$.

On generating samples of size 10 on computer, we observed samples of various types. We have solved equation (6.8.4) for $k = 1, 2,$ and solutions are represented by mme(1) and mme(2) respectively. For $k = 1$, equation (6.8.1) is given and for $k = 2$ equation (6.8.4) simplifies to

$$\frac{\sum_{i=1}^{n} x_i^2}{nx_o^2} = 2c^2 - \frac{1}{e^{1/c} - 1}(1+2c), \quad \text{where} \quad c = \frac{\sigma}{x_o} .$$

Table 6.8.1 gives σ/x_o as a function of $\bar{x}/x_o$ and Table 6.8.2 gives $\frac{\sigma}{x_o}$ as a function of $\frac{\Sigma x_i^2}{nx_o^2}$. These tables are analogous to a similar table for $k = 1$ given by Deemer and Votaw (1955), and can be used to obtain these estimators.

6.8.2 Comparison of various estimators

Now we consider the estimation of σ when sample contains a single outlier under the exchangeable model. Let $X_1,\ldots$ X_n be a sample of size n, out of which $(n-1)$ come from the distribution with pdf $f(x;\sigma)$ given at equation (6.1.2) and the

remaining one has a distribution with pdf $f(x; \frac{\sigma}{\alpha})$, $\alpha > 0$. For this purpose, we have compared U_1, U_2, U_3, U_4, U_{11} which had shown relatively better performance in untruncated exponential distribution with outliers. Also mme(1) and mme(2) discussed in Section 6.8.1 are included. We have also included one more estimator mme(3) which is the solution of the ml equation using U_4 in place of U_1.

For this situation, it is known that

$$E(U_1) = \frac{1}{n}[(n-1)E(X)+E(Y)] \text{ and } V(U_1) = \frac{1}{n^2}[(n-1)V(X)+V(Y)],$$

where E(X) and V(X) could be obtained from equation (6.8.3) and E(Y) and V(Y) could be evaluated by the same equation with σ replaced by σ/α. Finally, mse(U_1) is calculated. Here it could be noted that mse$(U_1)/\sigma^2$ is a function of x_o/σ, α and n. Similarly, mse's of other estimators considered in this section are evaluated and those are also the function of x_o/σ, α and n. Exact values of mse of first five estimators are evaluated for n = 5, 10, x_o/σ = 1, 2, 3, 4, 5, 10, ∞ and α = .1, .2, .5 and 1., using the moments calculated in Section 2.4. These values are given in Tables 6.8.3 and 6.8.4.

We have obtained simulated values of mse of all these estimators using 1000 iterations. In evaluation of mse of mme(1), mme(2) and mme(3), we have discarded those values of mme(1), mme(2) and mme(3) which are larger than 10 σ. The simulated values are given in Table 6.8.5. The simulated values of mse are more or less same to the exact values of mse whichever are available. This table reveals that for a fixed value

of x_o/σ if α changes then mse values of estimators are not much affected. This implies that a single outlier does not have much effect in the truncated exponential distribution when x_o is known. From the table it is also clear that mme(1), mme(2), and mme(3) are not good compared to other estimators. Table values shows that if x_o/σ is less than 4, U_1 is preferable to others and if x_o/σ is more than 4, we may use the procedure given for untruncated case in Chapter 4, i.e., we may use the estimator T. Now σ is the scale parameter of interest. So we follow the following procedure.

First calculate $\frac{x_o}{U_1}$ and if $\frac{x_o}{U_1} < 4$, use U_1 as an estimator of σ otherwise treat the sample from untruncated exponential distribution and estimate σ by the methods described in Chapter 4.

In Table 6.8.5, we have also given the proportion, in brackets, in 1000 samples where the mme(1), mme(2) or mme(3) do not exist. These proportions for all three decrease to 0 as $\frac{x_o}{\sigma} \to \infty$. There is not much dependence on α and the proportions for mme(1) and mme(2) are about equal. The mse's of U_1, U_2, U_3, U_4 and U_{11} for truncated case is much smaller than for the untruncated case even for outlier situation. This is in discrepancy with the conclusion drawn by Johnson and Kotz (1970, p. 223) where it is stated that "If $\frac{U_1}{x_o} > \frac{1}{2}$ then mme(1) which is mle of σ is infinity. This may be taken to mean that a truncated exponential distribution is inappropriate". But mme(1) is infinity for $\frac{U_1}{x_o} > \frac{1}{2}$ does not mean that this distribution is inappropriate. This only means that the mle does not perform good for truncated exponential distribution.

It could be seen from the Tables 6.8.3 and 6.8.4 that the mse's of all the estimators are minimum somewhere between $x_o/\sigma = 3$ and 4. So if one suspects that there is an outlier in a sample from an untruncated exponential distribution then one may even think of truncating the distribution at a reasonably large value since an outlier does not affect much for this case.

TABLE 6.7.1: Simulated values of biases of various estimators

p	α \ Estimator	T_{opt}	U_4/P_1	T_1	T_2	mle
0.80	0.10	3.174	-0.020	-0.054	-0.041	2.818
	0.20	1.218	-0.045	-0.078	-0.055	0.982
	0.50	0.189	-0.088	-0.132	-0.093	0.021
	1.00	-0.027	-0.137	-0.184	-0.137	-0.177
	2.00	-0.049	-0.183	-0.211	-0.174	-0.192
0.90	0.10	2.631	-0.022	-0.072	-0.054	2.377
	0.20	0.958	-0.043	-0.098	-0.071	0.749
	0.50	0.142	-0.099	-0.164	-0.113	-0.025
	1.00	-0.037	-0.156	-0.221	-0.166	-0.179
	2.00	-0.053	-0.199	-0.253	-0.203	-0.201
0.95	0.10	2.066	-0.020	-0.091	-0.065	1.934
	0.20	0.822	-0.061	-0.129	-0.089	0.671
	0.50	0.103	-0.104	-0.184	-0.125	-0.004
	1.00	-0.056	-0.172	-0.255	-0.193	-0.140
	2.00	-0.074	-0.219	-0.283	-0.228	-0.170
0.99	0.10	1.313	-0.027	-0.115	-0.083	1.370
	0.20	0.454	-0.077	-0.167	-0.119	0.447
	0.50	0.032	-0.139	-0.232	-0.156	0.050
	1.00	-0.062	-0.188	-0.277	-0.197	-0.034
	2.00	-0.103	-0.236	-0.316	-0.249	-0.081

TABLE 6.7.2: Simulated values of mse's of various estimators

p	α	T_{opt}	U_4/P_1	T_1	T_2	mle
0.80	0.10	18.004	0.057	0.034	0.034	14.932
	0.20	3.276	0.060	0.040	0.041	2.537
	0.50	0.180	0.072	0.056	0.056	0.121
	1.00	0.028	0.076	0.070	0.063	0.051
	2.00	0.031	0.085	0.080	0.073	0.059
0.90	0.10	13.369	0.072	0.049	0.051	11.768
	0.20	2.292	0.074	0.058	0.059	1.814
	0.50	0.146	0.077	0.072	0.071	0.100
	1.00	0.038	0.088	0.090	0.083	0.062
	2.00	0.042	0.097	0.106	0.097	0.069
0.95	0.10	9.143	0.086	0.066	0.071	8.823
	0.20	1.781	0.079	0.071	0.074	1.503
	0.50	0.135	0.091	0.090	0.088	0.097
	1.00	0.049	0.098	0.108	0.097	0.064
	2.00	0.054	0.108	0.125	0.113	0.072
0.99	0.10	4.090	0.091	0.086	0.090	4.866
	0.20	0.720	0.090	0.093	0.097	0.722
	0.50	0.107	0.097	0.112	0.107	0.105
	1.00	0.072	0.106	0.130	0.118	0.079
	2.00	0.081	0.123	0.152	0.136	0.083

TABLE 6.8.1: Table of σ/x_o as a function of $\bar{x}/x_o$

$\bar{x}/x_o$	σ/x_o	$\bar{x}/x_o$	σ/x_o
.01	.010	.25	.278
.02	.020	.26	.295
.03	.030	.27	.312
.04	.040	.28	.331
.05	.050	.29	.352
.06	.060	.30	.374
.07	.070	.31	.399
.08	.080	.32	.425
.09	.090	.33	.455
.10	.100	.34	.488
.11	.110	.35	.525
.12	.120	.36	.566
.13	.130	.37	.614
.14	.141	.38	.670
.15	.151	.39	.735
.16	.162	.40	.813
.17	.173	.41	.908
.18	.184	.42	1.026
.19	.196	.43	1.176
.20	.208	.44	1.377
.21	.221	.45	1.657
.22	.234	.46	2.075
.23	.248	.47	2.772
.24	.263	.48	4.163

TABLE 6.8.2: Table of σ/x_o as a function of $\Sigma x_i^2/nx_o^2$

$\Sigma x_i^2/nx_o^2$	σ/x_o	$\Sigma x_i^2/nx_o^2$	σ/x_o	$\Sigma x_i^2/nx_o^2$	σ/x_o
.005	.050	.155	.388	.301	2.537
.010	.071	.150	.403	.302	2.619
.015	.087	.165	.419	.303	2.707
.020	.100	.170	.436	.304	2.801
.025	.112	.175	.453	.305	2.902
.030	.123	.180	.472	.306	3.010
.035	.134	.185	.492	.307	3.126
.040	.144	.190	.513	.308	3.251
.045	.153	.195	.535	.309	3.386
.050	.163	.200	.559	.310	3.533
.055	.172	.205	.585	.311	3.694
.060	.181	.210	.613	.312	3.869
.065	.190	.215	.643	.313	4.061
.070	.199	.220	.675	.314	4.273
.075	.208	.225	.711	.315	4.509
.080	.217	.230	.749	.316	4.771
.085	.227	.235	.791	.317	5.065
.090	.236	.240	.838	.318	5.400
.095	.246	.245	.890	.319	5.780
.100	.256	.250	.948	.320	6.217
.105	.266	.255	1.013	.321	6.725
.110	.276	.260	1.086	.322	7.321
.115	.287	.265	1.171	.323	8.042
.120	.298	.270	1.268	.324	8.916
.125	.310	.275	1.382	.325	9.981
.130	.321	.280	1.517	.326	11.378
.135	.334	.285	1.680	.327	13.217
.140	.346	.290	1.880	.328	15.673
.145	.360	.295	2.132	.329	20.053
.150	.373	.300	2.459	.330	24.612

TABLE 6.8.3: Exact values of mse of various estimators for n=5

α	Est \ x_0/σ	1	2	3	4	5	10	∞
.1	U_1	.33779	.12355	.11212	.17718	.26852	.88067	7.40000
	U_2	.41993	.18499	.11695	.11924	.14998	.46278	4.66666
	U_3	.46639	.23442	.14879	.12586	.12877	.27255	2.79759
	U_4	.44523	.24349	.18851	.18270	.18926	.21365	.22714
	U_{11}	.36005	.17378	.16488	.20274	.24202	.33044	.37760
.2	U_1	.33968	.12693	.11315	.17068	.24845	.67199	1.80000
	U_2	.42170	.18918	.12179	.12199	.14729	.35927	1.05556
	U_3	.46813	.23881	.15489	.13217	.13351	.22787	.62840
	U_4	.44729	.24787	.19329	.18634	.19112	.20871	.21442
	U_{11}	.36222	.17736	.16618	.19916	.23274	.30127	.32280
.5	U_1	.34532	.13706	.11712	.15572	.20222	.32658	.36000
	U_2	.42700	.20163	.13636	.13173	.14503	.20288	.22222
	U_3	.47332	.25188	.17301	.15135	.14972	.17232	.18271
	U_4	.45351	.26124	.20846	.19925	.20025	.20545	.20580
	U_{11}	.36875	.18841	.17151	.19199	.21155	.23819	.23980
1.	U_1	.35456	.15318	.12545	.14475	.16700	.19909	.20000
	U_2	.43566	.22111	.15856	.14903	.15317	.16616	.16667
	U_3	.48184	.27241	.20006	.17945	.17588	.17885	.17911
	U_4	.46387	.28328	.23396	.22314	.22139	.22149	.22149
	U_{11}	.37957	.20696	.18369	.19127	.19885	.20556	.20562

TABLE 6.8.4: Exact values of mse of various estimators for n=10

α	Est \ x_o/σ	1	2	3	4	5	10	∞
.1	U_1	.33815	.10942	.06259	.07624	.10496	.26980	1.90000
	U_2	.38271	.14581	.07635	.06804	.07869	.17874	1.42975
	U_3	.40378	.16733	.09002	.07234	.07433	.13749	1.10230
	U_4	.39770	.17802	.11249	.09762	.09641	.10235	.10563
	U_{11}	.35197	.13766	.08956	.08949	.09830	.12024	.13117
.2	U_1	.33911	.11130	.06402	.07559	.10062	.21766	.50000
	U_2	.38363	.14792	.07876	.06967	.07845	.14797	.35537
	U_3	.40471	.16952	.09280	.07493	.07597	.11883	.27377
	U_4	.39875	.18035	.11528	.10018	.09844	.10228	.10353
	U_{11}	.35305	.13978	.09150	.09033	.09772	.11459	.11944
.5	U_1	.34196	.11689	.06835	.07456	.09089	.13137	.14000
	U_2	.38640	.15414	.08585	.07480	.07929	.10150	.10744
	U_3	.40748	.17600	.10099	.08268	.08160	.09399	.09820
	U_4	.40189	.18734	.12376	.10831	.10552	.10594	.10599
	U_{11}	.35627	.14617	.09753	.09361	.09761	.10431	.10466
1.	U_1	.34663	.12558	.07508	.07516	.08408	.09955	.10000
	U_2	.39093	.16379	.09630	.08272	.08336	.09061	.09090
	U_3	.41202	.18608	.11298	.09374	.09054	.09352	.09373
	U_4	.40706	.19847	.13691	.12107	.11734	.11621	.11621
	U_{11}	.36157	.15636	.10727	.10024	.10117	.10280	.10282

TABLE 6.8.5: Simulated values of mse of mme(1), mme(2), mme(3) and $\bar{X}$ for n = 10 and for x_o = 1,2,3,4,5, 10 and α = .1, .2, .5, 1.*

α	Est \ x_o/σ	1.	2.	3.	4.	5.	10.
.1	mme(1)	1.6812 (.233)	1.8833 (.083)	1.2326 (.022)	.7369 (.004)	.3609 (.000)	.3306 (.000)
	mme(2)	1.9056 (.227)	1.5024 (.086)	1.0892 (.026)	.6781 (.009)	.5335 (.000)	.6430 (.000)
	mme(3)	1.5073 (.041)	.9746 (.005)	.6132 (.003)	.2752 (.000)	.1395 (.000)	.1091 (.000)
	U_1	.3369	.1091	.0666	.0792	.1037	.2863
.2	mme(1)	1.8397 (.200)	1.9908 (.064)	1.4444 (.012)	.5721 (.003)	.3391 (.000)	.2822 (.000)
	mme(2)	2.1334 (.205)	1.7739 (.068)	1.3002 (.012)	.8304 (.002)	.4303 (.001)	.4737 (.000)
	mme(3)	1.2308 (.032)	.8489 (.011)	.4939 (.002)	.2755 (.000)	.1414 (.000)	.1142 (.000)
	U_1	.3442	.1120	.0610	.0809	.0944	.2395
.5	mme(1)	1.9531 (.204)	1.7665 (.072)	.8254 (.015)	.4998 (.003)	.3267 (.000)	.1248 (.000)
	mme(2)	1.9759 (.207)	12.7094 (.076)	.8699 (.017)	.6104 (.003)	.3055 (.001)	.1609 (.000)
	mme(3)	.8261 (.043)	.6505 (.009)	.1041 (.001)	.1835 (.001)	.1486 (.000)	.1018 (.000)
	U_1	.3451	.1155	.0667	.0728	.0927	.1165
1.	mme(1)	1.6148 (.224)	1.4616 (.056)	.8914 (.011)	.3750 (.002)	.1871 (.001)	.1152 (.000)
	mme(2)	1.6707 (.228)	1.5917 (.057)	1.1699 (.011)	.3490 (.003)	.1912 (.001)	.1270 (.000)
	mme(3)	1.4823 (.046)	.8101 (.007)	.2922 (.002)	.1631 (.002)	.1541 (.000)	.1177 (.000)
	U_1	.3462	.1243	.0753	.0726	.0804	.1092

* The values given in brackets are the simulated values of proportion for which mme(1), mme(2) and mme(3) do not exist in 1000 iterations.

REFERENCES

Abramowitz, M. and Stegun, I.A. (1965). "Handbook of Mathematical Functions", Dover Publications, Inc., New York.

Arnold, B.C. (1977). "Recurrence relations between expectations of functions of order statistics", Scand. Actuarial J., 169-174.

Bain, L.J. (1978). "Statistical Analysis of Reliability and Life-Testing Models", Marcel Dekker, Inc., New York and Basel.

Bain, L.J. and Weeks, D.L. (1964). "A note on the truncated exponential distribution", Ann. Math. Statist. 35: 1366-1367.

Bain, L.J., Engelhardt, M. and Wright, F.T. (1977). "Inferential procedures for the truncated exponential distribution", Commun. Statist.-Theor. Meth. 6(2): 103-111.

Balakrishnan, N. (1986). "Order statistics from discrete distributions", Commun. Statist.-Theor. Meth. 15(3): 657-675.

Balakrishnan, N. (1988). "Relations and identities for the moments of order statistics from a sample containing a single outlier", Commun. Statist.-Theor. Meth. 17(7): 2173-2190.

Balakrishnan, N. and Ambagaspitiya, R.S. (1988). "Relationships among moments of order statistics in samples from two related outlier models and some applications", Commun. Statist.-Theor. Meth. 17(7): 2327-2341.

Balakrishnan, N. and Joshi, P.C. (1984). "Product moments of order statistics from doubly truncated exponential distribution", Naval Res. Logist. Quart. 31: 27-31.

Balakrishnan, N. and Malik, H.J. (1985). "Some general identities involving order statistics", Commun. Statist.-Theor. Meth. 14(2): 333-339.

Balakrishnan, N., Malik, H.J. and Ahmed, S.E. (1988). "Recurrence relations and identities for moments of order statistics, II: Specific continuous distributions", Commun. Statist.-Theor. Meth. 17(8): 2657-2694.

Bapat, R.B. and Beg, M.I. (1989). "Order statistics for non-identically distributed variables and permanents", Sankhyā, Ser. A, Vol. 51, Part 1, 1989, 79-93.

Barnett, V. (1966). "Order statistics estimators of the location of the Cauchy distribution", J. Amer. Statist. Assn. 61: 1205-1218.

Barnett, V. and Lewis, T. (1984). "Outliers in Statistical Data", John Wiley and Sons, New York.

Chikkagoudar, M.S. and Kunchur, S.H. (1980). "Estimation of the mean of an exponential distribution in the presence of an outlier", The Canadian J. Statist. 8(1): 59-63.

Cohen, A.C., Jr. (1955). "Maximum likelihood estimation of the dispersion parameter of a chi-distributed radial error from truncated and censored samples with applications to target analysis", J. Amer. Statist. Ass. 50: 1122-1135.

David, H.A. (1981). "Order Statistics", John Wiley and Sons, New York.

David, H.A. and Shu, V.S.(1978). "Robustness of location estimators in the presence of an outlier", In: David, H.A. (Ed.), Contributions to survey sampling and applied statistics, Papers in honor of H.O. Hartley, 235-250, Academic Press, New York.

Deemer, W.L. and Votaw, D.F., Jr. (1955). "Estimation of parameters of truncated or censored exponential distributions", Ann. Math. Statist. 26: 498-504.

Epstein, B. and Sobel, M. (1953). "Life testing", J. Amer. Statist. Ass., Vol. 48: 486-502.

Epstein, B. and Sobel, M. (1954). "Some theorems relevant to life testing from an exponential distribution", Ann. Math. Statist., Vol. 25: 373-381.

Feller, W. (1972). "An Introduction to Probability Theory and Its Applications", Vol. 1, 3rd Edition, Wiley Eastern Private Limited, New Delhi.

Gastwirth, J.L. and Cohen, M.L. (1970). "Small sample behaviour of some robust linear estimators of location", J. Amer. Statist. Ass. 65: 946-973.

Gather, U. (1986). "Robust estimation of the mean of the exponential distribution in outlier situation", Commun. Statist.-Theor. Meth. 15(8): 2323-2345.

Gather, U. and Kale, B.K. (1988). "Maximum likelihood estimation in the presence of outliers", Commun. Statist.-Theor. Meth. 17(11): 3767-3784.

Govindarajulu, Z. (1963). "On moments of order statistics and quasi-ranges from normal populations", Ann. Math. Statist. 34: 633-651.

Gross, A.J., Hunt, H.H. and Odeh, R.E. (1986). "The correlation coefficient between the smallest and largest observations when (n-1) of the n observations are iid exponentially

distributed", Commun. Statist.-Theor. Meth. 15(4): 1113-1123.

Johnson, N.L. and Kotz, S. (1970). "Continuous Univariate Distributions-1", Houghton Mifflin Company, Boston.

Joshi, P.C. (1971). "Recurrence relations for the mixed moments of order statistics", Ann. Math. Statist. 42: 1096-1098.

Joshi, P.C. (1972). "Efficient estimation of the mean of an exponential distribution when an outlier is present", Technometrics 14: 137-143.

Joshi, P.C. (1973). "Two identities involving order statistics", Biometrika 60: 428-429.

Joshi, P.C. (1978). "Recurrence relations between moments of order statistics from exponential and truncated exponential distributions", Sankhyā, Ser. B, 39: 362-371.

Joshi, P.C. (1982). "A note on the mixed moments of order statistics from exponential and truncated exponential distributions", Journal of Statistical Planning and Inference 6: 13-16.

Joshi, P.C. (1988). "Estimation and testing under exchangeable exponential model with a single outlier", Commun. Statist.-Theor. Meth. 17(7): 2315-2326.

Joshi, P.C. and Balakrishnan, N. (1981). "Applications of order statistics in combinatorial identities", Journal of Combinatories, Information and System Sciences 6(3): 271-278.

Joshi, P.C. and Balakrishnan, N. (1982). "Recurrence relations and identities for the product moments of order statistics", Sankhyā 44, Ser. B, 39-49.

Kale, B.K. (1975). "Trimmed means and the method of maximum likelihood when spurious observations are present", Applied Statistics, R.P. Gupta (Ed.), 177-185, North Holland Publishing Company, Amsterdam.

Kale, B.K. and Sinha, S.K. (1971). "Estimation of expected life in the presence of an outlier observation", Technometrics, 13(4): 755-759.

Khan, A.H., Yakub, Mohd. and Parvez, S. (1983). "Recurrence relations between moments of order statistics", Naval Res. Logist. Quart. 30: 419-441.

Kimber, A.C. (1983). "Comparison of some robust estimators of scale in gamma samples with known shape", J. Statist. Comput. Simul. 18: 273-286.

Malik, H.J., Balakrishnan, N. and Ahmed, S.E. (1988). "Recurrence relations and identities for moments of order

statistics, I: arbitrary continuous distribution", Commun. Statist.-Theor. Meth. 17(8): 2623-2655.

Proschan, F. (1963). "Theoretical expectations of observed decreasing failure rate", Technometrics 5: 375-383.

Ranganathan, J. (1981). "Contributions to Inference in Exponential Models", Ph.D. Thesis, University of Poona, Pune.

Rao, C.R. (1974). "Linear Statistical Inference and Its Applications", John Wiley and Sons, New York.

Robson, D.S. and Whitlock, J.H. (1964). "Estimation of truncation point", Biometrika 51, 1 and 2, 33-39.

Saleh, A.K. Md. E. (1966). "Estimation of the parameters of the exponential distribution based on optimum order statistics in censored samples", Ann. Math. Statist. 37: 1717-1735.

Saleh, A.K. Md. E., Scott, Christine and Junkins, D.B. (1975). "Exact first and second order moments of order statistics from the truncated exponential distribution", Naval Res. Logist. Quart. 22: 65-77.

Schneider, H. (1986). "Truncated and Censored Samples from Normal Populations", Marcel Dekker, Inc., New York.

Vännman, K. (1976). "Estimators based on order statistics from a pareto distribution", J. Amer. Statist. Ass. 71: 704-708.

Vaughan, R.J. and Venables, W.N. (1972). "Permanent expressions for order statistic densities", J.R. Statist. Soc., Ser. B, 34: 308-310.

APPENDIX A

CALCULATIONS OF EXPECTED VALUES OF $X_{r:n}X_{s:n}$, $1 \leq r < s \leq n$ FOR TRUNCATED EXPONENTIAL DISTRIBUTION UNDER A SINGLE OUTLIER EXCHANGEABLE MODEL

For a given value of n, sample size, an algorithm is presented to compute the expected values of $X_{r:n}X_{s:n}$ for some particular value of r and s, where $1 \leq r < s \leq n$, when the sample contains one outlier. Expression for $\mu_{r,s:n}$ is given in equation (2.4.1). As explained in Section 2.4, each integral in $\mu_{r,s:n}$ is of the type

$$I(a,b,c,d,f) = \int_0^{x_0}\int_0^{y} xy(1-e^{-x})^a(e^{-x}-e^{-y})^b[e^{-y} - e^{-x_0}]^c\, e^{-xd}e^{-yf}\, dx\, dy$$

$$(A.1) \qquad = \int_0^{x_0}\int_0^{v} f(u,v)\, du\, dv \quad \text{(say)}.$$

We evaluate these integrals by using Gaussian quadrature.

Let a and b be finite real numbers then t point formula is

$$(A.2) \qquad \int_a^b f(y)\, dy \doteq \frac{(b-a)}{2}\sum_{i=1}^{t} w_i\, f(y_i),$$

where $y_i = \frac{(b-a)}{2}x_i + \frac{(b+a)}{2}, \quad i = 1,2,\ldots,t$

and the points x_i and weights w_i are extensively tabulated in Abramowitz and Stegun (1964) for various values of t. It should be noted that Equation (A.2) gives an exact result when f(y) is a polynomial of degree (2t-1) or less.

Now we apply equation (A.2) to the inner integral of equation (A.1), which gives

$$\int_0^{x_0}\int_0^{v} f(u,v)\,du\,dv = \int_0^{x_0} \frac{v}{2} \sum_{i=1}^{t} w_i\, f(\tfrac{v}{2}\,(x_i+1),\ v)\, dv$$

$$= \frac{1}{2} \sum_{i=1}^{t} w_i \int_0^{x_0} v\, f(\tfrac{v}{2}\,(x_i+1),\ v)\, dv\ .$$

Let $h(v) = v\, f(\frac{v}{2}(x_i+1), v)$ and using equation (A.2), we get

$$\int_0^{x_0}\int_0^{v} f(u,v)\,du\,dv \doteq \frac{1}{2} \sum_{i=1}^{t} w_i \frac{x_0}{2} \sum_{j=1}^{t} w_j\, h(\frac{(x_j+1)}{2}\, x_0)$$

$$= \frac{x_0}{4} \sum_{i=1}^{t} w_i \sum_{j=1}^{t} w_j \frac{x_0(x_j+1)}{2}$$

$$\cdot f(\frac{x_0(x_j+1)}{2}\ \frac{(x_i+1)}{2},\ \frac{x_0(x_j+1)}{2})$$

on substituting the value of h(.).

Using this approximation to each integral, $\mu_{r,s:n}$ can be obtained for $1 \le r < s \le n$. For given values of x_0, n, r, s, α, t, $\underset{\sim}{w}$ and $\underset{\sim}{x}$, the subprogram PMOM gives the $\mu_{r,s:n}$. The 10 point formula is used in this subroutine.

LANGUAGE

Fortran 10

STRUCTURE

SUBROUTINE PMOM (NN,IR,IS,FACT,ALP,XO,A,B,JC,JD,JE,X,W,U)

Formal parameters

NN	Integer	Input :	size of the sample for which moments have to be calculated.
IR	Integer	Input :	value of r of $X_{r:n}$.
IS	Integer	Input :	value of s of $X_{s:n}$.
FACT	Real	Input :	values of factorials.
ALP	Real	Input :	value of α.

XO	Real	Input :	truncation point of truncated exponential distribution.
A	Real	Input :	value of d described in equation (A.1).
B	Real	Input :	value of f described in equation (A.1).
JC	Integer	Input :	value of r-a, where a is given in equation (A.1).
JD	Integer	Input :	value of s-r-b, where b is given in equation (A.1).
JE	Integer	Input :	value of n-s-c, where c is given in equation (A.1).
X	Real	Input :	abscissas for 10 point Gaussian quadrature.
W	Real	Input :	weights for 10 points Gaussian quadrature
U	Real	Output:	expected value of $X_{r:n}X_{s:n}$.

Auxiliary algorithm

Subroutine PMOM calls subroutine ASUM.

This program is valid upto sample size 20. But these moments can be evaluated for larger n by giving more values of factorials.

```
      SUBROUTINE PMOM(NN,IR,IS,FACT,ALP,XO,A,B,JC,JD,JE,X,W,U)
C
C     SUBPROGRAM FOR CALCULATING E(X(IR)X(IS)) FOR A FIXED VALUE OF
C     IR,IS WHERE IR<IS
      DIMENSION A(8),B(8),FACT(20),U(20,20),JC(8),JD(8),JE(8),IC(8)
     1ID(8),IE(8),SUMM(40),X(10),W(10)
      FXO=1.-EXP(-XO)
      GXO=1.-EXP(-XO*ALP)
      NNN=NN-1
      DO 40 K=1,8
      IC(K)=IR-JC(K)
      ID(K)=IS-IR-JD(K)
      IE(K)=NN-IS-JE(K)
40    CONTINUE
      DO 10 N=1,8
      SUM=0.
      AA=A(N)
      BB=B(N)
      KC=IC(N)
      IF(KC.LT.O.) GO TO 5
      KD=ID(N)
      IF(KD.LT.0.) GO TO 5
      KE=IE(N)
      IF(KE.LT.0.) GO TO 5
      CALL ASUM(AA,BB,KC,KD,KE,X,W,XO,FXO,SUM)
5     SUMM(N)=SUM*XO/((FXO**(NN-1))*GXO*8.0)
10    CONTINUE
      T1=FLOAT(IR-1)*(SUMM(1)-SUMM(2))+FLOAT(NN-IS)*(GXO-1.)*SUMM(3)
      T2=T1+FLOAT(NN-IS)*SUMM(4)
      T4=T2+ALP*(SUMM(5)+SUMM(6))+FLOAT(IS-IR-1)*(SUMM(7)-SUMM(8))
      T3=FLOAT(IR)*FLOAT(IS-IR)*FLOAT(NN-IS+1)
      TOTAL=T3*T4
      DR=FACT(IR)*FACT(IS-IR)*FACT(NN-IS+1)
      U(IR,IS)=FACT(NN-1)*TOTAL/DR
      RETURN
      END
C
C     SUBPROGRAM FOR SUM
      SUBROUTINE ASUM(AA,BB,KC,KD,KE,X,W,XO,FXO,SUM)
      DIMENSION X(10),W(10)
      N1=10
      DO 20 I=1,N1
      DO 30 J=1,N1
      Z2=(X(J)+1.)*XO/2.
      Z1=Z2*(X(I)+1.0/2
      F1=Z1*Z2*((1.-EXP(-Z1))**KC)
      F2=(EXP(-Z1)-EXP(-Z2))**KD
      F3=(FXO-1.+EXP(-Z2))**KE
      F=F1*F2*F3*EXP(-AA*Z1)*EXP(-B8*Z2)
      AL=W(I)*W(J)*F*Z2*2.
      SUM=SUM+AL
30    CONTINUE
20    CONTINUE
      RETURN
      END
```

```
C
C                         CALLING PROGRAM
C
C     PROGRAM FOR PRODUCT MOMENTS TRUNCATED EXPONENTIAL DISTRIBUTION
C     BY USING N1(=t) POINT GAUSSIAN QUADRATURE FOR A PARTICULAR
C     VALUES OF xo, α, r, s AND n WHERE n ≤ 20
      DIMENSION A(8),B(8),FACT(20),X(10),W(10),JC(8),JD(8),JE(8)
      1U(20,20)
C
      DATA (X(I),I=1,10),(W(I),I=1,10),(JC(I),I=1,8),(JD(I),I=1,8),
     * (JE(I),I=1,8),
     * /.1488743390,.4333953941,.6794095683,.8650633667,.9739065285,
     * -.1488743390,-.4333953941,-.6794095683,-.8650633667,
     * -.9739065285,.2955242247,.2692667193,.2190863625,.1494513492,
     * .0666713443,.2955242247,.2692667193,.2190863625,.1494513492,
     * .0666713443,2,2,1,1,1,1,1,1,1,1,1,1,1,2,2,0,0,1,1,0,0,0,0
C
C     CALCULATION OF FACTORIALS
      FACT(1)=I.
      DO 70 J=2,20
   70 FACT(J)=FACT(J-1)*FLOAT(J)
C
      ALP=.2
      XO=2.
      A(1)=1.
      A(2)=1.+ALP
      A(3)=1.
      A(4)=1.
      A(5)=1.
      A(6)=ALP
      A(7)=ALP+1.
      A(8)=1.
      B(1)=1.
      B(2)=1.
      B(3)=1.
      B(4)=1.+ALP
      B(5)=ALP
      B(6)=1.
      B(7)=1.
      B(8)=1.+ALP
      NN=10
      IR=NN-1
      IS=NN
      CALL PMOM(NN,IR,IS,FACT,ALP,XO,A,B,JC,JD,JE,X,W,U)
      PRINT7,U(IR,IS)
    7 FORMAT(8X,F8.5/)
      STOP
      END
```

APPENDIX B

ROOTS OF MAXIMUM LIKELIHOOD EQUATIONS

An algorithm is presented to compute the roots of ml equations. As explained in Section 4.2, we use equations (4.2.2) and (4.2.4) which are

(B.1) $$n\bar{x} = (n - 1 + \frac{1}{\alpha})\sigma$$

(B.2) $$\sum_{i=1}^{n} (x_i - n\bar{x} + (n-1)\sigma)\, e^{\frac{x_i}{\sigma}\left(1-\frac{1}{n\bar{x}/\sigma-n+1}\right)} = 0$$

For given values of n and (x(I), I = 1,...,n), the subroutine ROOT gives maximum likelihood estimator of σ and α. If $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, it sets mle as $\bar{x}$. Otherwise it computes all the three roots of equation (B.2). For the second case, there will be one root $\bar{x}$ correspond to $\alpha = 1$. For other two roots of equation (B.2), we use Newton-Raphson method. But if this method fails to evaluate both the roots, then we evaluate the second root by using Bisection method.

Also if the root or roots calculated by Newton-Raphson method are not different from $\bar{x}$ or not within the interval

$$[\bar{x} - \frac{x_{(n)}-\bar{x}}{n-1},\ \bar{x} + \frac{(\bar{x}-x_{(1)})}{n-1}],$$

then we evaluate the root or roots by the Bisection method. Finally, mle of σ is that root out of all roots which maximizes the likelihood function. Then α_{mle} is evaluated using equation (B.1). This program is valid upto sample size 100.

APPENDIX B

ROOTS OF MAXIMUM LIKELIHOOD EQUATIONS

An algorithm is presented to compute the roots of ml equations. As explained in Section 4.2, we use equations (4.2.2) and (4.2.4) which are

$$\text{(B.1)} \qquad n\bar{x} = (n - 1 + \frac{1}{\alpha})\sigma$$

$$\text{(B.2)} \qquad \sum_{i=1}^{n} (x_i - n\bar{x} + (n-1)\sigma)e^{\frac{x_i}{\sigma}(1-\frac{1}{n\bar{x}/\sigma-n+1})} = 0$$

For given values of n and (x(I), I = 1,...,n), the subroutine ROOT gives maximum likelihood estimator of σ and α. If $\sum_{i=1}^{n} x_i^2 < (2n-1)\bar{x}^2$, it sets mle as $\bar{x}$. Otherwise it computes all the three roots of equation (B.2). For the second case, there will be one root $\bar{x}$ correspond to $\alpha = 1$. For other two roots of equation (B.2), we use Newton-Raphson method. But if this method fails to evaluate both the roots, then we evaluate the second root by using Bisection method.

Also if the root or roots calculated by Newton-Raphson method are not different from $\bar{x}$ or not within the interval

$$[\bar{x} - \frac{x_{(n)}-\bar{x}}{n-1}, \bar{x} + \frac{(\bar{x}-x_{(1)})}{n-1}] ,$$

then we evaluate the root or roots by the Bisection method. Finally, mle of σ is that root out of all roots which maximizes the likelihood function. Then α_{mle} is evaluated using equation (B.1). This program is valid upto sample size 100.

LANGUAGE

Fortran 10

STRUCTURE

SUBROUTINE ROOT(X,N,AML,ALP)

<u>Formal parameters</u>

X	Real	Input :	sample observations.
N	Integer	Input :	size of the sample.
AML	Real	Output:	maximum likelihood estimator of σ.
ALP	Real	Output:	maximum likelihood estimator of α.

```
      SUBROUTINE ROOT(X,N,AML,ALP)
C     SOLUTION OF LIKELIHOOD EQUATIONS BY ITERATIONS WHEN A SINGLE
C     OUTLIER IS PRESENT IN THE SAMPLE
      DIMENSION X(100),RS(3),EF(2),FF(2),IIF(2)
      AN=N
      AN1=AN-1.
      NN1=N-1.
      DO 64 K=1,NN1
      DO 54 I=K,NN1
      IF(X(K).LE.X(I+1)) GO TO 54
      TEMP=X(K)
      X(K)=X(I+1)
      X(I+1)=TEMP
 54   CONTINUE
 64   CONTINUE
C     THESE X'S ARE NOW ORDERED
      XT=0;XTT=0
      DO 44 I=1,N
      XT=XT+X(I)
 44   XTT=XTT+X(I)*X(I)
      XB=XT/AN
      IF[XTT.GT.(2n-1)*XB*XB] GO TO 2
      AML=XB
      ALP=1.
      RETURN
C
C     CALCULATIONS FOR FINDING THE ROOTS WHEN SUM OF SQUARES OF X(I)
C     IS MORE THAN (2*N-1)*(MEAN**2).WE ARE CALCULATING ONLY TWO
C     ROOTS BECAUSE ONE ROOT XB IS ALREADY KNOWN
C
 2    BLL=(XT-X(N))/AN1
      BUL=(XT-X(1))/AN1
      S1=BLL
C
C     NEWTON-RAPHSON METHOD
C
      DO 50 II=1,2
      IKK=0
 5    F1=0.
      F=0.
      IKK=IKK+1
      IF(IKK.GT.20) GO TO 20
      DO 10 I=1,N
      T=XT-AN1*S1
      H2=AN*X(I)*(XB/S1-1.)/T
      IF(ABS(H2).GE.55.) GO TO 1
      T2=EXP(H2)
      GO TO 15
 1    T2=0.
 15   F=F+(X(I)-T)*T2
      T1=AN*X(I)*(X(I)-T)*(-XB/(S1*S1*T)+(XB/S1-1.)*AN1/(T*T))
      F1=F1+(AN1+T1)*T2
 10   CONTINUE
      IF(F1.LE..1E-25) GO TO 20
      FF1=F/F1
```

```
      SO=S1-FF1
      IF(ABS(SO-S1).LE..00001) GO TO 25
      S1=SO
      GO TO 5
25    RS(II)=SO
C     IIF(II)=1 IMPLIES THAT NEWTON-RAPHSON METHOD FAILS IN
C     FINDING THE ROOT
      IIF(II)=0
      S1=(XT-2.*X(1))/AN1
      GO TO 50
20    IIF(II)=1
50    CONTINUE
C
      IF(IIF(1).EQ.1) GO TO 35
45    IF(IIF(2).EQ.1) GO TO 30
46    IF(ABS(RS(2)-RS(1)).LE..0001) GO TO 30
      IF(RS(2).GT.BUL) GO TO 30
      IF(RS(1).LT.BLL) GO TO 35
      IF(ABS(RS(1)-XB).LE..00001) GO TO 35
      IF(ABS(RS(2)-XB).LE..00001) GO TO 30
      GO TO 40
C
C     BISECTION METHOD
C
35    A1=BLL
      A2=XB
      E1=XT-AN1*A1
      E2=XT-AN1*A2
      E3=(E1-A1)/(E1*A1)
      E4=(E2-A2)/(E2*A2)
      IIK=0
      HA1=0.
      HA2=0.
      DO 55 I=1,N
      V1=X(I)*E3
      IF(ABS(V1).GT.55.) GO TO 60
      HA1=HA1+(X(I)-E1)*EXP(X(I)*E3)
      GO TO 55
60    HA1=HA1
55    CONTINUE
95    IF(IIK.GT.50) GO TO 65
      IF(ABS(HA1-HA2).LE..00001) GO TO 70
      IIK=IIK+1
75    C=(A1+A2)/2.
      HAC=0.
      DO 85 I=1,N
      E5=XT-AN1*C
      E8=X(I)*(E5-C)/(C*E5)
      IF(ABS(E8).GT.55.) GO TO 80
      HAC=HAC+(X(I)-E5)*EXP(X(I)*(E5-C)/(E5*C))
      GO TO 85
80    HAC=HAC
85    CONTINUE
      IF(HAC.LT.0.) GO TO 90
```

```
      A2=C
      HA2=HAC
      A1=A1
      GO TO 95
   90 A1=C
      HA1=HAC
      A2=A2
      GO TO 95
   70 IF(ABS(A1-A2).GE.0001) GO TO 75
   65 RS(1)=(A1+A2)/2.
      IIF(1)=0
      GO TO 45
   30 A2=BUL
      A1=XB
      E1=XT-AN1+A1
      E2=XT-AN1*A2
      E3=(E1-A1)/(E1*A1)
      E4=(E2-A2)/(E2*A2)
      IKK=0
      HA1=0.
      HA2=0.
      DO 105 I=1,N
      V2=X(I)*E4
      IF(ABS(V2).GT.55.) GO TO 110
      HA2=HA2+(X(I)-E2)*EXP(X(I)*E4)
      GO TO 105
  110 HA2=HA2
  105 CONTINUE
  115 IF (IKK.GT.50) GO TO 125
      IF(ABS(HA1-HA2).LE..00001) GO TO 130
      IKK=IKK+1
  135 C=(A1+A2)/2.
      HAC=0.
      DO 145 I=1,N
      E5=XT-AN1*C
      E8=X(I)*(E5-C)/(E5*C)
      IF(ABS(E8).GT.55.) GO TO 140
      HAC=HAC+(X(I)-E5)*EXP(X(I)*(E5-C)/(E5*C))
      GO TO 145
  140 HAC=HAC
  145 CONTINUE
      IF(HAC.LT.0.) GO TO 150
      A2=C
      HA2=HAC
      A1=A1
      GO TO 115
  150 A1=C
      HA1=HAC
      A2=A2
      GO TO 115
  130 IF(ABS(A1-A2).GE..0001) GO TO 135
  125 RS(2)=(A1+A2)/2.
      RS(3)=XB
      IIF(2)=0
      GO TO 46
```

```
C
C     CALCULATION OF LIKELIHOOD FUNCTION FOR BOTH THE ROOTS AND
C     HENCE MLE OF σ AND α
C
40    DO 155 I=1,2
      S2=RS(I)
      T4=1./(XT/S2-AN1)
      T5=1.-T4
      TT=0.
      DO 170 J=1,N
      YX=X(J)*T5)/S2
      IF(YX.LT.(-75)) GO TO 160
      TL=EXP(YX)
      GO TO 165
160   TL=0.
165   TT=TT+TL
170   CONTINUE
      YYY=XT/S2
      IF(YYY.LE.50.) GO TO 175
      IF(TT.LE..1E-10) GO TO 180
175   FF(I)=T4*EXP(-YYY)*TT/((S2)**N)
      GO TO 155
180   FF(I)=0.
155   CONTINUE
      IF(FF(1).LE.FF(2)) GO TO 100
      AML=RS(1)
      ALP=1/(N*XB/AML-AN+1.)
      RETURN
100   AML=RS(2)
      ALP=1/(N*XB/AML-AN+1.)
      RETURN
      END
C
```

A 108466

MATH-1989-D-RAN-EST